AL65-6264

Translation Number

B/162

ADDITIONAL COPIES AVAILABLE AT REDUCED RATES FROM

A TRANSLATION BY

MORRIS D. FRIEDMAN, INC.

Offices at
1383A WASHINGTON STREET

Mail to
P.O. BOX 35

WEST NEWTON 65, MASSACHUSETTS

Telephone

WOODWARD 9-8918 - 8919

MORRIS D. FRIEDMAN, INC.- 1383A Washington Street -West Newton 65, Massachusetts
Russian Translation

B-162

Characters of Linear Representations of Finite Groups on an Arbitrary Field

S. D. BERMAN

Matem. Sb. vol. 44(86), No. 4, 1958, pp.409-456

INTRODUCTION

The present paper is devoted to the theory of characters of linear representations of finite groups on an arbitrary field.

The author's results, published in [8] and [9], are generalized in §1. The concept of the Φ -character of a finite group is introduced in this same section.

An S -mapping of a finite group G is such a one-to-one mapping of a φ group into itself as defines the substitution in the set of continuously irreducible characters of this group. The class of S -mappings of a finite group G is exhausted by those one-to-one mappings of a group into itself as transform the classes of conjugate elements into each other and, hence, induce an automorphism of the algebras of these classes.

The Φ character of a group G, where Φ is a group of S -mappings of this group, is defined as the sum of different characters obtained from the absolutely irreducible character χ by the effect of all transformations from the group Φ on it.

Relations which generalize the classical dependences between absolutely irreducible characters of a finite group are proved for Φ characters. Here, Φ divisions of a group, sets of elements of the form $c^{-1}\varphi(a)c$ where $a \in G$ is a fixed element, c runs through the group G and φ is a group of S mappings of Φ, play the part of classes of conjugate elements.

Relations between the characters of a group G on an arbitrary field K', whose characteristic does not divide the order of the group, are obtained from general relations between Φ characters if Φ is taken to be a group of S -mappings of the form $a \to a^{\mu}$ $(a \in G)$, where μ runs through the integers corresponding to the automorphisms $\varepsilon \to \varepsilon^{\mu}$ of a field $K'(\varepsilon)$ on K' (ε is a primitive n-th root of 1, n is the least common multiple of the orders of the elements of the group G). In this case we call the Φ divisions of the group G, the K' -divisions of the group.

If G is a normal divisor of the group F and Φ is a group of S mappings of the group G generated by a group of inner automorphisms of the group F , then the Φ characters of G agree with the relative characters of the group G with respect to F , introduced by Frobenius [3].

Also considered in §1 are applications of the results on Φ characters to the question of the isomorphism of centers of group algebras.

§2 is devoted to induced representations. Two known theorems of R. Brauer on induced representations on an algebraically closed field of characteristic zero are generalized to representations on an arbitrary field K' of zero characteristic.

Let us call the K'-characters of a group G the characters of linear representations of a group G on a field K'. We shall call integer linear combinations of characters of irreducible representations of a group G on a field K' (irreducible K'-characters) generalized K'-characters.

Let us agree to call a subgroup $E \subseteq G$ satisfying the following conditions a K'-elementary subgroup of the group G :

1) E is a semi-direct product of a cyclic normal divisor $H = (a)$ of order h by the p-group $F(p,h) = 1$.

2) For any element $g \in F$, we will have $g^{-1}ag = a^{\mu}$, where μ is an integer such that the mapping $\varepsilon \rightarrow \varepsilon^{\mu}$ defines the automorphism of a field $K'(\varepsilon)$ on K' (ε is a primitive m-th root of 1 , m is the order of the group G).

The following equivalent statements are proved:

I. Each irreducible K'-character of a group G is represented as an integer linear combination of K'-characters induced by irreducible K'-characters of K'-elementary subgroups of the group G .

II. The function $f(g)$ $(g \in G)$, prescribed on the group G , with values in the field $K'' \supseteq K'$ is a generalized K'-character of this group if and only if $f(g)$ is a function of the K'-divisor of the group G and the function $f(g)$ induces a generalized K'-character of this subgroup on each elementary subgroup of the group G .

The method of [7] is used to prove statements I and II .

Furthermore, a theorem reciprocal to THEOREM II (a result of [13] is generalized) is established as is a theorem on the unsolvability of an integral ring of characters of a finite group in a direct sum of ideals. Relations are also studied between the Φ-characters of a group and a subgroup, which are analogous to the known Frobenius relations.

A brief explanation of the fundamental results of §2 was published in [10].

The following notation will be used in the sequel:

(A)

G - finite group

h - the order of G

n - the least common multiple of the orders of the elements of the group G

$C_1,\dots,C_s$ - classes of conjugate elements of the group G

h_i - order of the class C_i $(i = 1,\dots,s)$

K' - an arbitrary field whose characteristic does not divide h

$\hat{K}$ - algebraic closure of K'

K - field of complex numbers

R - field of rational numbers

$\mathfrak{p}$ - prime ideal of a field $R(\varepsilon)$, dividing the prime p

ε - primitive n-th root of 1

$R(G,K')$ - group algebra of the group G on the field K'

$\Gamma_1',\dots,\Gamma_r'$ - all irreducible representations of the group G on K'

$\hat{\Gamma}_1,\dots,\hat{\Gamma}_s$ - all irreducible representations of the group G on $\hat{K}$

$\Gamma_1,\dots,\Gamma_s$ - all irreducible representations of the group G on K

$\chi_i'(g)$ - character of the representations Γ_i' $(i = 1,\dots,r)$

$\hat{\chi}_j(g)$ - character of the representations $\hat{\Gamma}_j$ $(j = 1,\dots,s)$

$\chi_j(g)$ - character of the representations Γ_j $(j = 1,\dots,s)$

n_i - degree of the representation Γ_i $(i = 1,\dots,s)$

$Z_{K'}$ - center of the algebra $R(G,K')$

k_i - sum of elements of the class C_i in $R(G,K')$

$\hat{e}_1,\dots,\hat{e}_s$ - complete system of minimum idempotents of the center of $R(G,\hat{K})$

§1. Φ-CHARACTERS AND CHARACTERS OF REPRESENTATIONS ON AN ARBITRARY FIELD

Let G be a finite group, K' an arbitrary field.

It is assumed throughout this paper that the characteristic of the field K' does not divide the order of G.

The group algebra $R(G,K')$ decomposes into the direct sum of minimum, mutually-cancelling two-sided ideals:

$$R(G,K') = I_1' + \dots + I_r' \qquad (I_i' \cdot I_j' = 0 \text{ , if } i \neq j) \tag{1.1}$$

The expansion of the center $Z_{K'}$ of the algebra $R(G,K')$ into the following direct sum of fields corresponds to such an expansion:

(2.1) $$Z_{K'} = Z'_1 + \dots + Z'_r$$

(Z'_i is the center of the ideal I'_i ; $Z'_i \cdot Z'_j = 0$ if $i \neq j$).

Each field Z'_i $(i = 1,\dots,r)$ is isomorphically a subfield of the field $K'(\varepsilon)$.

Because of (1.1), for unit $R(G,K')$ there holds an expansion into a sum fo pairwise orthogonal minimum idempotents of the center:

(1'.1) $$1 = e'_1 + \dots + e'_r \quad (e'_i \in Z'_i \ ; \ e'_i \cdot e'_j = 0 \ \text{ for } \ i \neq j)$$

A two-sided ideal I'_i $(i = 1,\dots,r)$ decomposes into the direct sum of minimum left ideals:

(3.1) $$I'_i = I_{i1} + \dots + I_{is_i} \qquad (i = 1,\dots,r)$$

In conformance with (3.1), the minimum idempotent of the center e'_i $(i = 1,\dots,r)$ is represented as the sum of minimum idempotents of $R(G,K')$:

(3'.1) $$e'_i = e_{i1} + \dots + e_{is_i} \qquad (i = 1,\dots,r;\ e_{ij} \in I_{ij};\ e_{ij} \cdot e_{ik} = 0 \ \text{ for } \ j \neq k\ )$$

Each minimum two-sided ideal I'_i is isomorphic to the complete matrix ring on the body D_i , a finite extension of the field K' . The field Z'_i is the center of the body D_i $(i = 1,\dots,r)$.

According to general theorems of algebra [1], the dimension of the body D_i on its center Z'_i is m_i^2 , where m_i is an integer called the index of the body D_i $(i = 1,\dots,r)$.

The number of nonequivalent irreducible representations of the group G on a field K' is r . To each two-sided ideal I'_i $(i = 1,\dots,r)$ there corresponds an irreducible representation Γ_i' , on K' , of the group G which is defined by any minimum left ideal

$$I \subseteq I'_i \qquad (i = 1,\dots,r)$$

An arbitrary left ideal $I \trianglelefteq R(G,K')$ can be considered as an additive group with a ring of left operators of $R(G,K')$. I is a minimum if and only if the ring of operator endomorphisms of an additive group I is a body.

All minimum left ideals $I \subseteq I'_i$ $(i = 1,\dots,r)$ are operator isomorphic. The body D_i $(i = 1,\dots,r)$ is inversely isomorphic to a body of operator endomorphisms of any of these ideals.

The irreducible representation Γ_i' , on K' , of the group G decomposes on the field $\hat{K}$ into a sum of absolutely irreducible representations:

(4'.1) $$\Gamma_i' = m_i(\hat{\Gamma}_{i1} + \dots + \hat{\Gamma}_{iq_i})$$

where each absolutely irreducible representation I_{ij} $(j = 1,\dots,q_i)$ enters into the expansion Γ_i' with the same multiplicity m_i , the index of the body D_i $(i = 1,\dots,r)$ [4]. The number m_i is called the Schur index of any of the representations $\hat{\Gamma}_{ij}$ $(j = 1,\dots,q_i)$ relative to the field K' .

The Schur index m_i is a divisor of the powers of the absolutely irreducible representations $\hat{\Gamma}_{ij}$ $(j = 1,\dots,q_i ; i = 1,\dots,r)$.

As R. Brauer showed [16], each representation of a group G on a field $\hat{K}$ is equivalent to a representation in the field $\Pi(\varepsilon)$, where Π is s simple subfield of the field $\hat{K}$.

Absolutely irreducible representations of the group G on a field of characteristics $p > 0$ can be obtained from irreducible representations on the field $R(\varepsilon)$.

If $\mathfrak{p}$ is a prime ideal of a field $R(\varepsilon)$ dividing p and T is a ring of $\mathfrak{p}$ -integers* of the field $R(\varepsilon)$, then each irreducible representation Γ_i $(i = 1,\dots,s)$ in the field $R(\varepsilon)$ is equivalent to the matrix representation $g \rightarrow ||\alpha_{jk}^{(i)}(g)||$ $(g \in G)$, where $\alpha_{jk}^{(i)} \in T$. If the elements $\alpha_{jk}^{(i)} \in T$ are replaced by the corresponding classes of residues mod $\mathfrak{p}$, then all the absolutely irreducible representations of the group G on a field of characteristics p can be obtained from the representations $\Gamma_1,\dots,\Gamma_s$.

Hence, a one-to-one correspondence is established between the characters $\chi_1,\dots,\chi_s$ of the irreducible representations of G on the field of complex numbers [these representations are realized in the field $R(\varepsilon)$] and the characters $\hat{\chi}_1,\dots,\hat{\chi}_s$ of the absolutely irreducible representations of G on a field of characteristics p since the character $\hat{\chi}_i(g)$ is obtained from the character $\chi_i(g)$ $(i = 1,\dots,s)$ by reduction mod $\mathfrak{p}$.

The characters $\hat{\chi}_1(g),\dots,\hat{\chi}_s(g)$ of the irreducible representations $\hat{\Gamma}_1,\dots,\hat{\Gamma}_s$ of the group G on the field $\hat{K}$ satisfy the following basic relations:

$$\text{(I)} \qquad \hat{\chi}_i(g)\hat{\chi}_j(g) = \sum_{m=1}^{s} \gamma_{ij}^{(m)} \hat{\chi}_m(g)$$

($\gamma_{ij}^{(m)}$ are non-negative integers)

$$\text{(II)} \qquad h_i h_j \hat{\chi}_t(g_i)\hat{\chi}_t(g_j) = n_t \sum_{m=1}^{s} \lambda_{ij}^{(m)} h_m \hat{\chi}_t(g_m)$$

($g_1,\dots,g_s$ is a system of representatives of classes $C_1,\dots,C_s$; $\lambda_{ij}^{(m)}$ are non-negative integers).

* The ring T consists of all elements $\frac{a}{b}$ $(b \neq 0)$, where a and b are integer elements of the field $R(\varepsilon)$ and $b \not\equiv 0 \pmod{\mathfrak{p}}$.

(III) $$\sum_{m=1}^{s} \hat{\chi}_m(a)\hat{\chi}_m(b^{-1}) = \begin{cases} 0 & \text{if } a \text{ and } b \text{ belong to different classes } C_i \text{ and } C_j \\ \frac{h}{h_i} & \text{if } a, b \in C_i \end{cases}$$

(IV) $$\sum_{m=1}^{s} h_m \hat{\chi}_i(g_m)\hat{\chi}_j(g_m^{-1}) = \begin{cases} 0 & \text{for } i \neq j \\ h & \text{if } i = j \end{cases} \quad (g_m \quad C_m ;\ m = 1,\dots,s)$$

The equality (IV) can also be written as follows

(IV') $$\sum_{g \in G} \hat{\chi}_i(g)\, \hat{\chi}_j(g^{-1}) = \begin{cases} 0 & \text{if } i \neq j \\ h & \text{if } i = j \end{cases}$$

From (III) for $b = 1$, we obtain

($\widetilde{\text{III}}$) $$\sum_{m=1}^{s} n_m \hat{\chi}_m(a) = \begin{cases} 0 & \text{if } a \neq 1 \\ h & \text{if } a = 1 \end{cases}$$

If $\chi_j(g) = 1$, then (IV') becomes

(IV) $$\sum_{g \in G} \hat{\chi}_i(g) = \begin{cases} 0 & \text{if } \hat{\chi}_i(g) \neq 1 \\ h & \text{if } \hat{\chi}_i(g) \equiv 1 \end{cases}$$

The center $Z_{\hat{K}}$ of the group algebra $R(G,\hat{K})$ is an algebra on $\hat{K}$, which decomposes into the sum of s pairwise orthogonal fields isomorphic to the field $\hat{K}$. The elements $k_1,\dots,k_s$ and the idempotents $\hat{e}_1,\dots,\hat{e}_s$ [see the notation (A)] form two bases of the algebra $Z_{\hat{K}}$ on $\hat{K}$.

The following formulas hold:

(4.1) $$k_i = h_i \sum_{m=1}^{s} \frac{\hat{\chi}_m(g_i)}{n_m} \hat{e}_m \ ; \qquad \hat{e}_i = \frac{n_i}{h} \sum_{m=1}^{s} \hat{\chi}_i(g_m^{-1}) k_m$$
$$(g_i \in C_i \ ; \ i = 1,\dots,s)$$

DEFINITION 1.1. The one-to-one mapping φ of the group G onto itself will be called an S-mapping of the group if for any character $\chi_i(g)$ $(i = 1,\dots,s)$ the function $\chi_i(\varphi(g))$ is also an absolutely irreducible character.

It evidently follows from the equality $\chi_i(\varphi(g)) = \chi_j(\varphi(g))$ $(g \in G)$ that $\chi_i(g) = \chi_j(g)$ for all $g \in G$. Therefore, the S-mapping φ defines a substitution in the set of irreducible character of the group G on the complex number field.

It is clear that if φ, ψ are S-mappings of the group G then φ^{-1}

and $\varphi\psi$ are also S-mappings.

REMARK 1.1. As has been noted, the characters $\hat{\chi}_1,\ldots,\hat{\chi}_s$ of the irreducible representations of the group G on an arbitrary algebraic closed field $\hat{k}$ whose characteristic does not divide the order of the group, are obtained from complex characters $\chi_1,\ldots,\chi_s$ by a reduction mod $\mathfrak{p}$.

Therefore, the S-mapping φ defines a one-to-one mapping of the set $\hat{\chi}_1,\ldots,\hat{\chi}_s$ into itself for any field $\hat{k}$.

DEFINITION 2.1. Let φ be an S-mapping of the group G. The linear transformation φ of the linear space of the group algebra $R(G,K')$, defined by the formula

$$\varphi\left(\sum_{i=1}^{h} \alpha_i g_i\right) = \sum_{i=1}^{h} \alpha_i \varphi(g_i) \qquad (\alpha_i \in K')$$

($g_1,\ldots,g_h$ are elements of G) will be called an S-mapping of the algebra $R(G,K')$ induced by the mapping φ of the group G.

In the sequel, we shall not distinguish between the S-mapping of the algebra $R(G,K')$ and that S-mapping of the group which induces this mapping.

Examples of S-mappings of a group:

1. Arbitrary automorphism φ of a group. Actually, if $\chi_i(g)$ is the character of an irreducible representation $g \to A(g)$ of the group G, then $\chi_i(\varphi(g))$ is the character of an irreducible representation $g \to A(\varphi(g))$ of this group.

2. The transformation $\varphi(g) = g^{\mu}$ $(g \in G)$, where $(\mu,h) = 1$. The transformation φ can be put into correspondence with the automorphism $\tilde{\varphi}$ of the Galois group of the field $R(\varepsilon)$ on R: $\tilde{\varphi}(\varepsilon) = \varepsilon^{\mu}$. Hence, the following formula is correct:

$$\tilde{\varphi}[\chi_i(g)] = \chi_i(g^{\mu}) \tag{5.1}$$

Actually, if $\chi_i(g) = \varepsilon^{q_1} + \ldots + \varepsilon^{q_r}$ (the values of each character are the sums of the roots of unity), then

$$\tilde{\varphi}[\chi_i(g)] = \varepsilon^{\mu q_1} + \ldots + \varepsilon^{\mu q_r}$$

On the other hand, $\chi_i(g^{\mu}) = \varepsilon^{\mu q_1} + \ldots + \varepsilon^{\mu q_r}$ since the eigennumbers of the matrix A^{μ} are μ-powers of the eigennumbers of the matrix A.

Because of (5.1), $\chi_i(\varphi(g))$ is the character of that representation of the group G which is obtained from the representation Γ_i corresponding to the character χ_i $(i = 1,\ldots,s)$ as a result of the automorphism $\tilde{\varphi}$ of the field of representations $R(\varepsilon)$.

LEMMA 1.1. Each S-mapping φ of the group G transforms the classes of conjugate elements of a group into each other.

PROOF. Let $a \in G$. If $\varphi(a)$ and $\varphi(g^{-1}ag)$ belong to different classes C_i for a certain element $g \in G$, then because of (III)

$$Q = \sum_{i=1}^{s} \chi_i[(\varphi(a))^{-1}]\chi_i(\varphi(g^{-1}ag)) = 0$$

But $\chi_i(\varphi(g^{-1}ag)) = \chi_i(\varphi(a))$ since the characters $\chi_i(\varphi(g))$ are functions of the class of conjugate elements. Therefore, on the basis of the same relation (III):

$$Q = \sum_{i=1}^{s} \chi_i[(\varphi(a))^{-1}]\chi_i(\varphi(a)) = \frac{h}{h_j}$$

where h_j is the order of the class $C_j \supseteq \varphi(a)$. Contradictions are obtained.

LEMMA 1'.1. If φ is a S-mapping of the group G , then $\varphi(1) = 1$.

PROOF. Let $\chi_i(\varphi(g)) = \chi_{r_i}(g)$ $(i = 1,\ldots,s)$. Then $\chi_i(\varphi(1)) = n_{r_i}$ (n_{r_i} is the power of the representation corresponding to the character χ_{r_i}). If $\varphi(1) \neq 1$, then because of (III)

$$\sum_{i=1}^{s} n_i n_{r_i} = \sum_{i=1}^{s} \chi_i(1)\chi_i(\varphi(1)) = 0$$

which is impossible.

COROLLARY. For any S-mapping φ the powers of the absolutely irreducible representations corresponding to the characters $\chi_i(g)$ and $\chi_i(\varphi(g)) = \chi_{r_i}(g)$ $(g \in G)$ coincide.

Actually, $n_i = \chi_i(1) = \chi_i(\varphi(1)) = \chi_{r_i}(1) = n_{r_i}$.

LEMMA 2.1. The S-mapping φ of the algebra $R(G,\hat{K})$ transforms the minimum idempotents of the center

(4''.1) $$\hat{e}_i = \frac{n_i}{h} \sum_{g \in G} \hat{\chi}_i(g)g$$

into each other.

Actually

(6.1) $$\varphi(\hat{e}_i) = \frac{n_i}{h} \sum_{g \in G} \hat{\chi}_i(g)\varphi(g) = \frac{n_i}{h} \sum_{g \in G} \hat{\chi}_i(\varphi^{-1}(g))g$$

Since $\hat{\chi}_i(\varphi^{-1}(g))$ is a character of the group G to which the absolutely irreducible representation of power n_i corresponds, by virtue of the corollary to LEMMA 1'.1, then we obtain, from a comparison of (4''.1) and (6.1), that

$\varphi(\hat{e}_i)$ is the minimum idempotent of the center $R(G,\hat{K})$.

THEOREM 1'.1. The one-to-one mapping φ of the group G into itself is an S -mapping of a group if and only if φ induces an automorphism of the center Z of the algebra $R(G,K)$ for which the elements k_i $(i = 1,\ldots,s)$ transform into one another.

PROOF. The substitution of idempotents $\begin{pmatrix} e_1 \ldots e_s \\ e_{r_1} \ldots e_{r_s} \end{pmatrix}$ corresponds to each automorphism ψ of the algebra Z on K and, conversely, any of the $s!$ substitutions of the elements $e_1,\ldots,e_s$ determines an automorphism of the algebra Z on K $(Z = Ke_1 + \ldots + Ke_s)$. Therefore, if φ induces an automorphism of the center, then

$$(6'.1) \qquad \varphi(e_i) = \frac{n_i}{h} \sum_{g \in G} \chi_i(\varphi^{-1}(g))g = \frac{n_{r_i}}{h} \sum_{g \in G} \chi_{r_i}(g)g$$

Comparing coefficients for 1 in the right and left sides of (6'.1), we obtain:

$$n_i = n_{r_i}$$

from which

$$\chi_i(\varphi^{-1}(g)) = \chi_{r_i}(g) \qquad (g \in G\ ;\ \ i = 1,\ldots,s)$$

Therefore φ is an S -mapping of the group G .

The necessity of the conditions of the theorem results from LEMMA 2.1 .

COROLLARY. If φ is an S -mapping of the group G and $a \in G$, then

$$(7.1) \qquad \varphi(a^{-1}) = g^{-1}[\varphi(a)]^{-1}g \qquad (g \in G)$$

PROOF: Let $\varphi(k_i) = k_{r_i}$ $(i = 1,\ldots.s)$. If

$$(7'.1) \qquad k_i k_t = \sum_{m=1}^{s} \lambda_{it}^m k_m$$

then because of THEOREM (1'.1)

$$(8.1) \qquad k_{r_i} k_{r_t} = \sum_{m=1}^{s} \lambda_{it}^m k_{r_m}$$

Let us assume that the elements k_j are enumerated so that $k_1 = 1$; a C_i ; a^{-1} C_t . Then, from (7'.1), the following equality results

$$\lambda_{it}^1 = h_i = h_t$$

[see the notation (A)] and (8.1) shows that the class k_{r_t} contains the element $[\varphi(a)]^{-1}$ (otherwise, the equality $\lambda_{it}^1 = 0$ must hold).

Naturally, the question occurs: Do S -mappings of a finite group exist which determine such a substitution of the set of classes of conjugate elements of this group as is not induced by some mapping of the form $\varphi\psi$, where φ is an automorphism of the group and ψ is a mapping of form $g \to g^{\mu}$ $[g \in G$; $(\mu,h) = 1]$. The following example yields an affirmative answer to this question.

Let us consider a group G of 36-th order with the defining relations: $a^3 = 1$, $b^3 = 1$, $ab = ba$, $c^2 = 1$, $c^{-1}ac = b^{-1}$, $c^{-1}bc = a^{-1}$, $d^2 = 1$, $d^{-1}ad = a^{-1}$, $d^{-1}bd = b^{-1}$, $cd = dc$.

The classes of conjugate elements of the group G are:

$$C_1 = \{1\}, \ C_2 = \{a,a^{-1},b,b^{-1}\}; \quad C_3 = \{ab,a^2b^2\}$$
$$C_4 = \{ab^2,a^2b\}, \ C_5 = \{cd,a^2bcd,ab^2cd\}, \ C_6 = \{acd,bcd,a^2b^2cd\}$$
$$C_7 = \{a^2cd,abcd,b^2cd\}, \ C_8 = \{c,ac,a^2c,bc,b^2c,abc,a^2bc,a^2b^2c\}$$
$$C_9 = \{d,ad,a^2d,bd,b^2d,abd,a^2bd,ab^2d,a^2b^2d\}$$

Let us show that a transposition of the classes (C_8,C_9) is an automorphism of the center Z of the algebra $R(G,K)$. Actually, the product C_iC_j, where $i, j < 8$, contains only elements of the classes C_q $(1 \le q < 8)$ and for the elements k_8 and k_9, corresponding to the classes C_8 and C_9, the following multiplication table holds:

$$k_8k_i = h_ik_8 \quad (i = 1,\dots,4); \qquad k_8k_j = 3k_9 \quad (j = 5,6,7)$$
$$k_8^2 = 9(k_1 + k_2 + k_3 + k_4); \qquad k_8k_9 = 9(k_5 + k_6 + k_7)$$
$$k_9k_i = h_ik_9 \quad (i = k,\dots,4); \qquad k_9k_j = 3k_8 \quad (j = 5,6,7); \qquad k_9^2 = 9(k_1+k_2+k_3+k_4)$$
$$(h_i \text{ is the order of the class } C_i)$$

to which the transposition (C_8,C_9) remains unchanged.

Let us assume that there exists an automorphism φ of the group G which induces the automorphism $\alpha = (C_8,C_9)$ of the center Z of the algebra $R(G,K)$. Then $\varphi(d^{-1}ad) = \varphi(a^{-1})$, from which $c^{-1}\varphi(a)c = [\varphi(a)]^{-1}$ because $\varphi(d) = a^ib^jc$. The equality $c^{-1}xc = x^{-1}$ is satisfied only for the elements ab, a^2b^2, a^2b and ab^2 of the group $(a)^x(b)$. This means $\varphi(a)$ is one of these elements; the latter is a contradiction that the automorphism φ remains at the place of the class $C_2 = \{a,a^{-1},b,b^{-1}\}$.

Since any mapping $g \to g^{\mu}$, $(\mu,36) = 1$, remains at the place of each class C_i $(i = 1,\dots,9)$, then we have thereby proved that no mapping of the form $\varphi\psi$ exists, where φ is an automorphism of the group G, $a\psi(g) = g^{\mu}$, $(\mu,36) = 1$ $(g \quad G)$, which induces the S-mapping $\alpha = (C_8,C_9)$ of the center Z.

LEMMA 3.1. Let R be a finite-dimensional, linear space on the field P; Φ a group of linear transformations of the space R; $M = \{u_1,\dots,u_m\}$ and $M' = \{u_1',\dots,u_m'\}$ are such two bases of R on P that, under the effect of the transformation $\varphi \in \Phi$, the vectors of each of the sets M and M' transform into each other. Let the sets M and M' decompose into nonintersecting subsets of mutually Φ -equivalent elements:

$$M = \bigcup_{i=1}^{q} M_i , \quad M_i \cap M_j = \Lambda \quad \text{for } i \neq j , \quad M_i = \left\{u_1^{(i)}, \ldots, u_{r_i}^{(i)}\right\} \quad (i=1,\ldots,q)$$

$$M' = \bigcup_{j=1}^{r} M'_j , \quad M'_i \cap M'_j = \Lambda \quad \text{if } i \neq j , \quad M'_j = \left\{u_1'^{(j)}, \ldots, u_r'^{(j)}\right\} (j=1,\ldots,r)$$

Then the vectors $v_1 = u_1^{(1)} + \ldots + u_{r_1}^{(1)}, \ldots, v_q = u_1^{(q)} + \ldots + u_{r_q}^{(q)}$ and $v'_1 = u_1'^{(1)} + \ldots + u_{q_1}'^{(1)}, \ldots, v'_r = u_1'^{(r)} + \ldots + u_{q_r}'^{(r)}$ form two bases of the subspace $\tilde{R} \subseteq R$ consisting of all vectors of the space R keeping each transformation $\varphi \in \Phi$ and, therefore, $r = q$.

If, above all, R is a semisimple commutative algebra on P and the basis $M = \{u_1, \ldots, u_m\}$ is a system of pairwise orthogonal minimum idempotents of the algebra R , then R is a sub-algebra of R .

PROOF: Let $x = \lambda_1 u_1 + \ldots + \lambda_j u_j + \ldots + \lambda_k u_k + \ldots \in \tilde{R}$ $(\lambda_i \in P)$. If the vectors u_k and u_j belong to one subset M_i , then there exists a transformation $\varphi \in \Phi$ such that $\varphi(u_j) = u_k$. Then
$\varphi(x) = \lambda_1 \varphi(u_1) + \ldots + \lambda_j u_k + \ldots$. On the other hand,
$\varphi(x) = x = \lambda_1 u_1 + \ldots + \lambda_k u_k + \ldots$; this means $\lambda_j = \lambda_k$.

Hence, the coefficients for the vectors of one set M_i in the expression for x agree, wherefore we conclude that x is represented as a linear combination: $x = \gamma_1 v_1 + \ldots + \gamma_q v_q$ $(\gamma_i \in P ; i = 1,\ldots,q)$. On the other hand, $v_i \in \tilde{R}$ $(i = 1,\ldots,q)$.. Therefore, the vectors $v_1,\ldots,v_q$ form a basis of $\tilde{R}$. In exactly the same manner, we find that $v'_1,\ldots,v'_r$ is a basis of $\tilde{R}$.

If R is an algebra over P and $u_1,\ldots,u_m$ are pairwise orthogonal idempotents of R , then

$$v_i v_j = \begin{cases} 0 & \text{for } i \neq j \\ v_i & \text{if } i = j \end{cases}$$

This means $(\gamma_1 v_1 + \ldots + \gamma_q v_q)(\beta_1 v_1 + \ldots + \beta_q v_q) = \gamma_1 \beta_1 v_1 + \ldots + \gamma_q \beta_q v_q \in \tilde{R}$ $(\gamma_i, \beta_i \in P)$, i.e., $\tilde{R}$ is a subalgebra of R .

The LEMMA is proved.

Let $X = \{\hat{\chi}_1, \ldots, \hat{\chi}_s\}$ be a set of characters of representations of G which are irreducible on K ; $Q = \{C_1, \ldots, C_s\}$ a set of classes of conjugate elements of the group G ; $M = \{\hat{e}_1, \ldots, \hat{e}_s\}$ a set of minimum idempotents of the center of $R(G,\hat{K})$; Φ is an arbitrary group of S -mappings of $R(G,\hat{K})$.

In view of LEMMAS 1.1 and 2.1 , Q and M decompose under the effect of transformations from Φ of the set X into non-intersecting regions of transitivity:

$$X = X_1 \cup \ldots \cup X_q \; ; \qquad M = M_1 \cup \ldots \cup M_q \; ; \qquad Q = Q_1 \cup \ldots \cup Q_r$$

where

(9.1) $X_1 = \{\hat{\chi}_{i1},\ldots,\hat{\chi}_{ir_i}\}$; $M_i = \{\hat{e}_{i1},\ldots,\hat{e}_{ir_i}\}$ $(i = 1,\ldots,q)$

$Q_j = \{C_{j1},\ldots,C_{jq_j}\}$ $(j = 1,\ldots,r)$

DEFINITION 3.1. The sets $X_1,\ldots,X_q$ will be called Φ divisions of the characters of G , the sets $M_1,\ldots,M_q$ are Φ divisions of minimum idempotents of the center of $R(G,K)$. The set-theoretical sum T_j of elements of the group G belonging to classes from the set Q_j $(j = 1,\ldots,r)$ will be called a Φ division of the group G . The characters of one Φ division of characters and elements of one Φ division of a group we shall agree to call Φ conjugates.

Evidently the elements $a, b \in G$ [the characters $\hat{\chi}_i(g)$ and $\hat{\chi}_j(g)$] are Φ conjugates if and only if there exists an S -mapping $\varphi \in \Phi$ such that $b = c^{-1}\varphi(a)c$ [correspondingly, $\hat{\chi}_j(g) = \hat{\chi}_i(\varphi(g))$].

On the basis of the corollary of LEMMA 1 , the powers of absolutely irreducible representations corresponding to the characters $\hat{\chi}_{i1},\ldots,\hat{\chi}_{ir_i}$ of one Φ division X_i $(i = 1,\ldots,q)$ coincide:

(10.1) $$n_{i1} = \ldots = n_{ir_i} \qquad (i = 1,\ldots,q)$$

The classes $C_{j1},\ldots,C_{jq_j}$ contained in one Φ division T_j $(j = 1,\ldots,r)$ also have the identical order.

Because of (6'.1) and (9.1), the Φ divisions of characters and idempotents define each other (φ^{-1} runs through the group Φ when φ runs through this group): if the character $\hat{\chi}_{ij} \in X_i$, then the idempotent

$$\hat{e}_{ij} = \frac{n_{ij}}{h} \sum_{g\in G} \hat{\chi}_{ij}(g^{-1})g \in M_i$$

The minimum idempotents $\hat{e}_1,\ldots,\hat{e}_s$ of the center $R(G,\hat{K})$ and the elements $k_1,\ldots,k_s$ [k_i is the sum of elements of the class C_i in $R(G,\hat{K})$] form two bases of the center $Z_{\hat{K}}$, satisfying the conditions of LEMMA 3.1. In conformance with (9.1), let us put

(10'.1) $t_j = k_{j1} + \ldots + k_{jq_j}$ $(j = 1,\ldots,r)$; $\tilde{e}_i = \hat{e}_{i1} + \ldots + \hat{e}_{ir_i}$ $(i=1,\ldots,q)$

The element t_j is the sum of elements of the Φ division T_j $(j = 1,\ldots,r)$ in $R(G,\hat{K})$.

Because of LEMMA 3.1, we obtain the statement:

THEOREM 1.1. The elements $t_1,\ldots,t_r$ and the idempotents $\tilde{e}_1,\ldots,\tilde{e}_q$ of the center $Z_{\hat{K}}$ of the algebra $R(G,\hat{K})$ form two bases of the subalgebra $\tilde{Z}_{\hat{K}} \subseteq Z_{\hat{K}}$ consisting of all elements of the center maintaining each transformation $\varphi \in \Phi$.

Hence, there results in particular that the number of Φ divisions of characters of the group G equals the number of Φ divisions of the group.

COROLLARY. The following formulas hold:

$$(11.1) \qquad t_i = \sum_{j=1}^{q} \alpha_{ij}\tilde{e}_j \, , \quad \tilde{e}_i = \sum_{j=1}^{q} \beta_{ij} t_j \quad (\alpha_{ij}, \beta_{ij} \in \hat{K} \, ; \, i,j = 1,\ldots,q; \; \tilde{e}_i\tilde{e}_j = 0 \text{ if } i \neq j)$$

$$(12.1) \qquad t_i t_j = \sum_{m=1}^{q} \lambda_{ij}^{(m)} t_m \qquad (\lambda_{ij}^{(m)} \text{ are non-negative integers})$$

Formulas (11.1) are a generalization of the equality (4.1). The relation (12.1) expresses the law of composition of Φ divisions of a group.

Let us note that because of (11.1) the following equality is satisfied for any element $x \in \tilde{Z}_{\hat{K}}$ and an arbitrary idempotent $\tilde{e}_i$ $(i = 1,\ldots,q)$:

$$(11'.1) \qquad x\tilde{e}_i = \lambda\tilde{e}_i \qquad (\lambda \in \hat{K} \, ; \, i = 1,\ldots,q)$$

DEFINITION 4.1. We shall call the idempotents $e \in \tilde{Z}_{\hat{K}}$, i.e., idempotents of a center maintaining the effect of all the transformations from the group Φ , the Φ idempotents of the center of $R(G,\hat{K})$. The idempotents $\tilde{e}_1,\ldots,\tilde{e}_q$ will be called the minimum Φ idempotents of the center.

We shall agree to call the character of a certain representation G on K (not absolutely irreducible) which remains in place under the effect of all transformations $\varphi \in \Phi$, a Φ character. The sum of characters of one Φ division of characters

$$(12'.1) \qquad \tilde{\chi}_i(g) = \hat{\chi}_{i1}(g) + \ldots + \hat{\chi}_{ir_i}(g) \qquad (i = 1,\ldots,q \, ; \, g \in G)$$

will be called an irreducible Φ character of G .

The representation G on $\hat{K}$ whose character is a Φ character, we shall call a Φ representation.

Since $\tilde{Z}_{\hat{K}}$ is a semisimple commutative algebra with a basis $\tilde{e}_1,\ldots,\tilde{e}_q$ then each Φ idempotent $e \in \tilde{Z}_{\hat{K}}$ is represented uniquely as the sum of certain of the minimum Φ idempotents $\tilde{e}_i$ $(i = 1,\ldots,q)$. The minimum Φ idempotents $\tilde{e}_1,\ldots,\tilde{e}_q$ are not representable as the sum of Φ idempotents of the center.

Because any representation of the group G over $\hat{K}$ is completely reducible, taking LEMMA 3 into account we arrive at the following conclusion:

LEMMA 4.1. Each Φ-character $\tilde{\chi}(g)$ on the field $\hat{K}$ is represented as an integer linear combination of irreducible Φ characters:

$$\tilde{\chi}(g) = \sum_i \alpha_i \tilde{\chi}_i(g) \qquad (\alpha_i \geq 0)$$

Evidently there exists a one-to-one correspondence between the irreducible Φ-characters on an arbitrary field $\hat{K}$ and the irreducible Φ-characters on the field of complex numbers K because of the same correspondence between characters of irreducible representations on these fields. (The transition from the Φ-character on the field K to the Φ-character on the field $\hat{K}$ of characteristics p is accomplished by reduction mod [see (A)].)

THEOREM 2.1. Each irreducible Φ-character $\tilde{\chi}_i(g)$ $(i = 1,\dots,q)$ is a function of the Φ-division of G :

(13.1) $$\tilde{\chi}_i(a) = \tilde{\chi}_i(b)$$

if the elements $a, b \in G$ are Φ-conjugate.

PROOF. Let $\tilde{\chi}_i(a) = \tilde{\chi}_{i1}(a) + \dots + \tilde{\chi}_{ir_i}(a)$, where $\tilde{\chi}_{i1},\dots,\tilde{\chi}_{ir_i}$ are all the characters of G irreducible over $\hat{K}$; Φ-conjugate to $\hat{\chi}_{i1}$; $b = \varphi(a)$ $(\varphi \in \Phi)$. Then

$\hat{\chi}_i(b) = \hat{\chi}_{i1}(\varphi(a)) + \dots + \hat{\chi}_{ir_i}(\varphi(a)) = \hat{\chi}_{i1}(a) + \dots + \hat{\chi}_{ir_i}(a) = \hat{\chi}_i(a)$ because characters from one Φ-division of characters transform into themselves under the effect of the S-mapping φ.

Relations similar to (I) - (IV) for the characters of absolutely irreducible representations can be established between the irreducible Φ-characters $\hat{\chi}_i$ of the group G .

Let us introduce the notation:

(B)
- l_i - the number of elements (orders) of the Φ-division T_i $(i = 1,\dots,q)$
- t_i - the sum of the elements of the Φ-division T_i in $R(G,\hat{K})$ $(i=1,\dots,q)$
- $b_1,\dots,b_q$ - the system of representations of the Φ-divisions $T_1,\dots,T_q$
- r_i - the number of absolutely irreducible characters in a Φ-division of the characters

$$X_i = \{\hat{\chi}_{i1}, \dots, \hat{\chi}_{ir_i}\} \qquad (i = 1,\dots,q)$$

- n_i - the degree of an absolutely irreducible representation corresponding to any of the characters $\hat{\chi}_{ij}$ $(j = 1,\dots,r_i;\ i = 1,\dots,q)$ [see (10.1)]

In deriving the relations between the Φ-characters, we will assume that the characteristic of the field $\hat{K}$ does not divide the numbers l_i and r_i $(i = 1,\ldots,q)$.

Because of (4.1), (10.1), (10'.1), (12'.1) and (13.1), the second of formulas (11.1) can be written as:

$$(11''.1) \qquad \tilde{e}_i = \frac{n_i}{h} \sum_{j=1}^{q} \tilde{\chi}_i(b_j^{-1})t_j = \frac{n_i}{h} \sum_{g\in G} \tilde{\chi}_i(g^{-1})g$$

Let us take two irreducible I-characters of the group G:

$$(14.1) \qquad \tilde{\chi}_i(g) = \hat{\chi}_{i1}(g) + \ldots + \hat{\chi}_{ir_i}(g) \quad \text{and} \quad \tilde{\chi}_j(g) = \hat{\chi}_{j1}(g) + \ldots + \hat{\chi}_{jr_j}(g)$$

The product $\tilde{\chi}_i(g)\cdot\tilde{\chi}_j(g)$ is evidently a Φ-character of G since for the S-transformation $\varphi \in \Phi$

$$\tilde{\chi}_i(\varphi(g))\cdot\tilde{\chi}_j(\varphi(g)) = \tilde{\chi}_i(g)\tilde{\chi}_j(g)$$

Therefore, because of LEMMA 4.1, we obtain

$$(I') \qquad \tilde{\chi}_i(g)\tilde{\chi}_j(g) = \sum_{k=1}^{q} \gamma_{ij}^{(k)} \tilde{\chi}_k(g) \qquad (\gamma_{ij}^{(k)} \text{ are non-negative integers})$$

(the first relation between the irreducible Φ-characters).

By virtue of (14.1)

$$\sum_{g\in G} \tilde{\chi}_i(g)\tilde{\chi}_j(g^{-1}) = \sum_{m=1}^{r_i} \sum_{t=1}^{r_j} \left(\sum_{g\in G} \hat{\chi}_{im}(g)\hat{\chi}_{jt}(g^{-1}) \right)$$

As a consequence of (IV')

$$\sum_{g\in G} \hat{\chi}_{im}(g)\hat{\chi}_{jt}(g^{-1}) = \begin{cases} 0 & \text{if } (i,m) \neq (j,t) \\ h & \text{if } (i,m) = (j,t) \end{cases}$$

This means that

$$(IV'') \qquad \sum_{g\in G} \tilde{\chi}_i(g)\tilde{\chi}_j(g^{-1}) = \begin{cases} 0 & \text{if } i \neq j \\ hr_i & \text{for } i = j \end{cases}$$

(the fourth relation between the irreducible Φ-characters).

Because of (13.1), formula (IV'') can be written as:

$$(IV''') \qquad \sum_{k=1}^{q} l_k\tilde{\chi}_i(b_k^{-1})\tilde{\chi}_j(b_k) = \begin{cases} 0 & \text{if } i \neq j \\ hr_j & \text{if } i = j \end{cases}$$

Let us put

$$(15.1) \qquad \alpha_{ik} = \tilde{\chi}_i(b_k^{-1}) \; ; \qquad \delta_{kj} = \frac{\tilde{\chi}_j(b_k)l_k}{h\cdot r_j} \qquad (i,j,k = 1,\ldots,q)$$

Then (IV''') shows that between the matrices $||\alpha_{ij}||$ and $||\delta_{ij}||$ there exists the dependence:

$$(16.1) \qquad ||\alpha_{ij}||^{-1} = ||\delta_{ij}||$$

Therefore

$$\sum_{j=1}^{q} \delta_{kj}\alpha_{ji} = \begin{cases} 0 & \text{if } k \neq i \\ 1 & \text{if } k = i \end{cases}$$

(III')
$$\sum_{j=1}^{q} \frac{1}{r_j} \tilde{\chi}_j(b_k)\, \tilde{\chi}_j(b_i^{-1}) = \begin{cases} 0 & \text{if } i \neq k \\ \dfrac{h}{l_k} & \text{if } i = k \end{cases}$$

(the third relation between irreducible Φ-characters).

By virtue of (11''.1) and (15.1)

$$\frac{h}{n_i}\tilde{e}_i = \sum_{m=1}^{q} \tilde{\chi}_i(b_m^{-1})t_m = \sum_{m=1}^{q} \alpha_{im}t_m$$

Then because of (16.1) and (15.1)

(17.1)
$$t_k = \sum_{m=1}^{q} \delta_{km}\left(\frac{h}{n_m}\tilde{e}_m\right) = \sum_{m=1}^{q} \frac{l_k}{h\cdot r_m}\cdot\frac{h}{n_m}\tilde{\chi}_m(b_k)\tilde{e}_m = l_k \sum_{m=1}^{q} \frac{\tilde{\chi}_m(b_k)}{n_m \cdot r_m}\tilde{e}_m$$

from which, in view of the orthogonality of the idempotents $\tilde{e}_1,\ldots,\tilde{e}_q$, we obtain:

(18.1)
$$t_k\tilde{e}_j = \frac{l_k}{n_j r_j}\tilde{\chi}_j(b_k)\tilde{e}_j$$

Multiplying both sides of (12.1) by $\tilde{e}_k$ and taking (18.1) into account, we arrive at the relation:

$$\frac{l_i l_j}{(n_k r_k)^2}\tilde{\chi}_k(b_i)\,\tilde{\chi}_k(b_j) = \sum_{m=1}^{q} \frac{\lambda_{ij}^{(m)}\, l_m}{n_k r_k}\tilde{\chi}_k(b_m)$$

or

(II')
$$l_i l_j \tilde{\chi}_k(b_i)\,\tilde{\chi}_k(b_j) = n_k r_k \sum_{m=1}^{q} \lambda_{ij}^{(m)}\, l_m \tilde{\chi}_k(b_m)$$

(second relation between irreducible Φ-characters).

If the group of S-mappings of Φ is such that each of the Φ-characters $\tilde{\chi}_i(g)$ coincides with the absolutely irreducible character $\tilde{\chi}_i(g)$ $(i = 1,\ldots,s)$ then each Φ-division T_i of the group G evidently coincides with the class of conjugate elements C_i $(i = 1,\ldots s)$ and formulas (I') - (IV') transform, respectively, into the relations (I) - (IV) between the absolutely irreducible characters. In this case, formulas (11''.1) and (17.1) do not differ from (4.1).

LEMMA 5.1. Let Φ be a group of mutually one-to-one mappings of the finite set M; Φ' the normal divisor of Φ; $Q_1,\ldots,Q_s$ nonintersecting subsets of Φ'-equivalent elments into which M decomposes under the effect of a transformation from Φ'. Then any transformation $\varphi \in \Phi$ transforms the subsets $Q_1, \ldots, Q_s$ into each other.

PROOF. If $a, b \in Q_i$, then there exists a transformation $\psi \in \Phi'$ such that $b = \psi(a)$. Since Φ' is a normal divisor of the group Φ then $\varphi\psi = \psi_1\varphi$, where $\psi_1 \in \Phi'$. This means $\varphi(b) = \varphi\psi(a) = \psi_1\varphi(a)$, i.e., the elements $\varphi(a)$ and $\varphi(b)$ are Φ'-equivalent and belong to a certain subset Q_j. Hence, $\varphi(Q_i) \subseteq Q_j$. In exactly the same manner we obtain $\varphi^{-1}(Q_j) \subseteq Q_i$. Therefore, $\varphi(Q_i) = Q_j$.

The LEMMA is proved.

It is possible to consider mutually one-to-one mappings of the group G into itself which play the same part with respect to Φ-characters and Φ-divisions as do the S-mappings with respect to the absolutely irreducible characters and classes of conjugate elements of the group.

DEFINITION 5.1. A mutually one-to-one mapping φ of the group G into itself will be called and S-Φ-mapping if:

1) φ determines a mutually one-to-one mapping of the set of Φ-divisions of the group G into itself.*

2) For any irreducible Φ-character $\tilde{\chi}(g)$ of a group G over a field of complex numbers, the function $\tilde{\chi}(\varphi(g))$ is also an irreducible Φ-character.

If φ is a S - Φ - mapping of G, then by virtue of the mutually one-to-one correspondence between irreducible Φ-characters of the group G over the field of complex numbers K and over an arbitrary closed algebraic field $\hat{K}$, 2) is also satisfied for Φ-characters over $\hat{K}$.

An example of an S - Φ-transformation is the S-mapping $\varphi \in N(\Phi)$, where $N(\Phi)$ is the normalizer of the subgroup Φ in the group of all mutually one-to-one mappings of G into itself.

Actually, as a consequence of LEMMA 5.1. such a transformation φ determines the substitutions in the sets of Φ-divisions of G and the irreducible Φ-characters of G, i.e., conditions 1) and 2) are satisfied.

In particular, the arbitrary S-mapping $g \to g^{\mu}$ $(g \in G, (\mu,h) = 1)$ is a S - Φ - mapping with respect to any group Φ, consisting of the transformation $g \to g^{\nu}$ $((\nu,h) = 1)$ since the group of all such mappings is abelian (it is isomorphic to the multiplicative group of classes of residues mod n containing numbers mutually prime to n (see the notation (A)).

* It can be shown, exactly as for S-mappings (see LEMMA 1.1) that condition 1) results from condition 2).

If a group F of $S - \Phi$ -transformations of the group G is given, then the set of Φ-characters of G $\{\tilde{\chi}_1, \ldots, \tilde{\chi}_q\}$ and the set $\{T_1, \ldots, T_q\}$ of Φ-divisions of G decompose into an identical number of nonintersecting transitivity domains under the effect of transformations from F.

This is easily proved on the basis of LEMMA 3.1 and formula (11.1) by the same means as was used to obtain THEOREM 1.1 by means of the same LEMMA and formula (4.1).

DEFINITION 6.1. Let us agree to say that an irreducible Φ-character $\tilde{\chi}_i(g)$ (the Φ-division T_j of the group G) sustains an $S - \Phi -$ transformation of φ, if $\tilde{\chi}_i(\varphi(g)) = \tilde{\chi}_i(g)$ for all $g \in G$ (correspondingly, $\varphi(g) \in T_j$ if $g \in T_j$).

THEOREM 3.1. The number of irreducible Φ-characters of the group G maintaining the $S - I$ -transformation φ equals the number of I-divisions of the group sustaining this transformation.

PROOF. Let $T_1, \ldots, T_q$ be Φ-divisions of G, $b_i \in T_i$ $(i = 1, \ldots, q)$; $\tilde{\chi}_1(g), \ldots, \tilde{\chi}_q(g)$ irreducible Φ-characters of G; α the number of Φ-divisions of G sustaining the transformation φ; β the number of Φ-characters $\tilde{\chi}_i(g)$ remaining in place under the effect φ.

Since β is independent of the algebraic, closed field $\hat{K}$ on which the representations G are taken, then we take the field K of complex numbers as the field of representations.

Let us compute the sum by two methods:

$$D = \sum_{k=1}^{q} \sum_{j=1}^{q} \frac{l_j}{r_k} \tilde{\chi}_k(\varphi(b_j))\tilde{\chi}_k(b_j^{-1}) \qquad \text{(see notation (B))}$$

Because of (III')

$$D = \sum_{j=1}^{q} \left(\sum_{k=1}^{q} \frac{l_j}{r_k} \tilde{\chi}_k(\varphi(b_j))\, \tilde{\chi}_k(b_j^{-1}) \right) = h\alpha$$

and as a consequence of (IV''')

$$D = \sum_{k=1}^{q} \frac{1}{r_k} \left(\sum_{j=1}^{q} l_j \tilde{\chi}_k(\varphi(b_j))\tilde{\chi}_k(b_j^{-1}) \right) = h\beta$$

This means $\alpha = \beta$. The theorem is proved.

THEOREM 3.1 is a generalization of the Frobenius-Schur theorem [3].

The number of characters of the group G sustaining the transformation $g \to g^{\mu}$ equals the number of classes of conjugate elements sustaining this transformation.

DEFINITION 7.1. Let $F = (\varphi)$ be a cyclic group of $S - \Phi$ transformations of the group G. Let us agree to say that an irreducible Φ-character $\tilde{\chi}_i(g)$ (a Φ-division T_j of the group G) belongs to the transformation φ^k, if φ^k is a generating element of the subgroup $F' \subseteq F$, consisting of all transformations $\psi \in F$, which the Φ-character $\tilde{\chi}_i(g)$ (correspondingly, the Φ-division T_j) maintains.

LEMMA 6.1. If $F = (\varphi)$ is a cyclic group of $S - \Phi$ transformations of the group G, then the number of irreducible Φ-characters of the group G which belong to the transformation φ^k equals the number of Φ-divisions of the group belonging to this transformation.

PROOF. For $k = 1$ the statement of the theorem is valid since the irreducible Φ-character $\tilde{\chi}_i(g)$ (correspondingly, the Φ-division T_j of the group G) belongs to φ if and only if it maintains this transformation.

Let us assume that the LEMMA is correct for all $k < s$ ($1 \leq s \leq n'$; n' is the order of F) and let us prove it for $k = s$.

The Φ-characters (Φ-divisions) maintaining the transformation φ^s evidently belong to the transformations φ^k, where $0 < k \leq s$ and $s \equiv 0 (\mathrm{mod}\ k)$.

Let $s, s_1, \ldots, s_m$ be all positive divisors of s; β_i (β_i') the number of irreducible Φ-characters (correspondingly Φ-divisions of G) belonging to φ^{s_i} ($i = 1, \ldots, m$); γ (γ') the number of Φ-characters $\tilde{\chi}_i(g)$ (correspondingly, Φ-divisions T_j) belonging to φ^s; δ (δ') the number of irreducible Φ-characters (correspondingly Φ-divisions of G) maintaining the transformation φ^s.

The following equalities hold:

$$\delta = \gamma + \beta_1 + \ldots + \beta_m; \qquad \delta' = \gamma' + \beta_1' + \ldots + \beta_m'$$

by virtue of the assumptions of induction, $\beta_i = \beta_i'$ ($i = 1, \ldots, m$) and, $\delta = \delta'$ because of THEOREM 3.1. This means $\gamma = \gamma'$.

The LEMMA is proved.

THEOREM 4.1. The $S - \Phi$-transformations of a set of irreducible Φ characters and Φ-divisions of G decompose, under the effect of a cyclic group F, into an identical number of regions of transitivity, where the appropriate regions of transitivity of these sets contain the same number of elements.

PROOF. Let $M_1, \ldots, M_t$ ($M_1', \ldots, M_t'$) be subsets of elements mutually equivalent, into which the set of irreducible Φ-characters of G (correspondingly, the set of Φ-divisions of G) is decomposed under the effect of transformations from F.

Let us assume that n' is the order of the group $F = (\varphi)$; $\alpha_1,\ldots,\alpha_k$ is such a system of positive divisors of n' that each irreducible Φ-character (Φ-division of G) belongs to one of the transformations φ^{α_i}, $(i = 1,\ldots,k)$.

The Φ-character $\tilde{\chi}_j(g)$ (correspondingly, the Φ-division T_j of the group G) belongs to the transformation φ^{α_i} if and only if the number of elements in the set $M_q \supseteq \tilde{\chi}_j$ (correspondingly, in the set $M'_s \supseteq T_j$) equals α_i (α_i is the index in F of the subgroup (φ^{α_i}) which maintains $\tilde{\chi}_j(T_j)$ in place).

If m_i irreducible Φ-characters belong to the transformation φ^{α_i}, then just as many Φ-divisions of G belong to this transformation because of LEMMA 6.1.

Therefore, there exists exactly $\frac{m_i}{\alpha_i}$ subsets M_j containing α_i elements and the same conclusion would be valid for the set M'_j.

The THEOREM is proved.

THEOREM 4.1 is a strengthening of THEOREM 4.1 for the case of a cyclic group F of $S - \Phi$-transformations.

If the group F is not cyclic, then THEOREM 4.1 is not true, as can easily be shown by examples.

Let us consider the application of the results obtained to the theory of representations of finite groups on an arbitrary field. Let us use the notations (A) and (B).

Let K' be an arbitrary field whose characteristic does not divide the order h of the finite group G; $\hat{K}$ the algebraic closure of K'; n the least common multiple of the orders of the elements of G; ε the primary root of degree n of 1; F the Galois group of the field $K'(\varepsilon)$ on K'. Each automorphism $\psi \in F$ is given by the formula

(19.1) $$\psi(\varepsilon) = \varepsilon^{\nu}; \quad (\nu,n) = 1$$

In conformance with the group F, let us substitute its isomorphic group Φ consisting of the S-transformation $g \to g^{\nu}$ $(g \in G)$, where ν is an integer such that the transformation $\varepsilon \to \varepsilon^{\nu}$ defines the automorphism $\psi \in F$.

DEFINITION 8.1. Let Φ be a group of S -mappings of G corresponding to the Galois group of the field $K'(\varepsilon)$ on K' (the group F). Let us call the Φ -divisions of the group G , the characters of G and the minimum idempotents of the center $R(G,\hat{K})$, respectively, the K' -divisions of G , the K' -divisions of the characters of G and the K' -divisions of the minimum idempotents of the center $R(G,\hat{K})$.

The characters of one K' -division of the characters, the idempotents of one K' -division of the idempotents and the elements of one K' -division of the group G will be called K' conjugate

The characters of the representations of G on K' (traces of the representation matrix) will be called K' -characters. The characters $\chi_i'(g)$ $(i = 1,\ldots,r)$ of representations Γ_i' $(i = 1,\ldots,r)$ of the group G which are irreducible on K' will be called irreducible K' -characters. We shall retain the terminology character for the traces of matrices of absolutely irreducible representations.

Let $X_i = \{\hat{\chi}_{i1},\ldots,\hat{\chi}_{ir_i}\}$ $(i = 1,\ldots,q)$ be K' -divisions of characters of the group G ; $E_i = \{\hat{e}_{i1},\ldots,\hat{e}_{ir_i}\}$ $(i = 1,\ldots,q)$ the corresponding K' -divisions of the minimum idempotents of the center $R(G,\hat{K})$; $T_1,\ldots,T_q$ the K' -divisions of G ; l_i the order of T_i $(i = 1,\ldots,q)$; $b_1,\ldots,b_q$ is a subset of the elements of G such that $b_i \in T_i$ $(i = 1,\ldots,q)$. If $\hat{e}_j = \frac{n_j}{h} \sum_{g\in G} \hat{\chi}_j(g^{-1})g$ [see (4.1)] are minimum idempotents of the center $R(G,\hat{K})$ and $\varphi \in \Phi$ is an S -mapping corresponding to the automorphism $\psi \in F$ given by formula (19.1), then because of (6.1) and (5.1)

$$\varphi(e_j) = \frac{n_j}{h} \sum_{g\in G} \hat{\chi}_j(\varphi^{-1}(g^{-1}))g = \frac{n_j}{h} \sum_{g\in G} \psi^{-1}[\hat{\chi}(g^{-1})]g$$

Hence, the effect of the S -mapping φ on $\hat{e}_j$ reduces to the effect of the automorphism $\psi^{-1} \in F$ on the coefficient of this idempotent.

Since each idempotent $e \in Z_{\hat{K}}$ ($Z_{\hat{K}}$ is the center $R(G,\hat{K})$) is represented as the sum of certain of the minimum idempotents $\hat{e}_j$, then it is hence easy to conclude that $e \in R(G,K')$ if and only if e is a Φ -idempotent, i.e., maintains the effect of all transformations $\varphi \in \Phi$ (the element $u \in K'(\varepsilon)$ in this and only this case belongs to the fundamental field K' when $\psi(u) = u$ for all automorphisms $\psi \in F$).

Therefore, the minimum idempotents of the center $R(G,K')$ coincide with the minimum Φ-idempotents of the center $R(G,\hat{K})$ which means they are given by the formulas:

$$(20.1) \qquad e_i' = \hat{e}_{i1} + \ldots + \hat{e}_{ir_i} = \frac{n_i}{h} \sum_{g \in G} \tilde{\chi}_i(g^{-1})g$$

where the Φ - character $\tilde{\chi}_i(g)$ equals:

$$(20'.1) \qquad \tilde{\chi}_i(g) = \hat{\chi}_{i1}(g) + \ldots + \hat{\chi}_{ir_i}(g) \qquad (i = 1,\ldots,q)$$

(n_i is the degree of the absolutely irreducible representation corresponding to any of the characters $\hat{\chi}_{ij}$ $(j = 1,\ldots,r_i)$; see (11''.1)).

Each minimum idempotent e_i' $(i = 1,\ldots,q)$ of the center $R(G,K')$ generates a minimum two-sided ideal I_i' in $R(G,K')$ which is isomorphic to the complete matrix ring on the set D_i with index m_i . The representation Γ_i' of the group G $(i = 1,\ldots,q)$, which is irreducible on K' , corresponds to this ideal.

Since q , by virtue of THEOREM 1.1 , is the number of K'-divisions of G , then the following theorem holds [8]:

THEOREM 5.1. The number of irreducible representations of a finite group G on an arbitrary field K' whose characteristic does not divide the order of G equals the number of K'-divisions of the group G .

There results from (20.1) and (20'.1) that the representation Γ_i' decomposes into a sum of absolutely irreducible representations corresponding to the characters $\hat{\chi}_{i1},\ldots,\hat{\chi}_{ir_i}$ $(i = 1,\ldots,q)$. Hence, according to (4'.1), each of these representations enters into the decomposition of Γ_i' with the multiplicity m_i $(i = 1,\ldots,q)$. Hence, the irreducible K'-characters $\chi_i'(g)$ of the representations Γ_i' $(i = 1,\ldots,q)$ are expressed by the formula:

$$(21.1) \qquad \hat{\chi}_i'(g) = m_i(\hat{\chi}_{i1}(g) + \ldots + \hat{\chi}_{ir_i}(g)) = m_i\hat{\chi}_i(g) \qquad (i = 1,\ldots,q)$$

from which

$$(22.1) \qquad \tilde{\chi}_i(g) = \frac{\chi_i'(g)}{m_i} \qquad (i = 1,\ldots,q)$$

(m_i divides n_i , which means it is not divided by the characteristic of the field $\hat{K}$).

Substituting the Φ-character $\tilde{\chi}_i(g)$ (formula (22.1)) into relations (II'), (III'), (IV') , we obtain the following fundamental relations between the characters of the irreducible representations of a group on the field K' :

$$(II'') \qquad l_i l_j \chi_k'(b_j)\chi_k'(b_j) = n_k r_k m_k^2 \sum_{t=1}^{q} \lambda_{ij}^{(t)} l_t \chi_k'(b_t)$$

(the second fundamental relation between the irreducible K' -characters);

(III'')
$$\sum_{j=1}^{q} \frac{1}{r_j m_j^2} \chi_j'(b_i)\chi_j'(b_k^{-1}) = \begin{cases} 0 & \text{if } i \neq k \\ \frac{h}{l_i} & \text{if } i = k \end{cases}$$

(the third fundamental relation between the irreducible K' -characters);

(IV'')
$$\sum_{g \in G} \chi_i'(g)\chi_j'(g^{-1}) = \begin{cases} 0 & \text{if } i \neq j \\ h r_i m_i^2 & \text{if } i = j \end{cases}$$

(the fourth fundamental relation between the irreducible K' -characters).

The first fundamental relation between irreducible K' -characters is obtained by starting from the expansion formula of the direct product of Γ_i' and Γ_j' into irreducible representations: $\Gamma_i' \times \Gamma_j' = \sum_{k=1}^{q} \alpha_{ij}^{(k)} \Gamma_k'$ from which

(I'')
$$\chi_i'(g)\chi_j'(g) = \sum_{k=1}^{q} \alpha_{ij}^{(k)} \chi_k'(g)$$

(the first fundamental relation between irreducible K' -characters).

Let us note that relations (II') and (III'), just exactly as the corresponding formulas for the Φ -characters, are valid under the assumption that the orders of the K' -divisions l_i and the numbers r_i $(i = 1,\ldots,q)$ are not divided by the characteristic of the field K' .

Substituting the Φ -character $\tilde{\chi}_i(g)$ given by (22.1) into (I'), we arrive at the equality:

(I''')
$$\chi_i'(g)\chi_j'(g) = \sum_{k=1}^{q} \frac{\gamma_{ij}^{(k)} m_i m_j}{m_k} \chi_k'(g)$$

Comparing (I'') and (I'''), we obtain the relation

(23.1)
$$\gamma_{ij}^{(k)} = \frac{\alpha_{ij}^{(k)} \cdot m_k}{m_i m_j}$$

The following proposition results from (23.1):

Let K' be a field of characteristics zero; Γ_i' a representation of the group G with the character χ_i' , which is irreducible on K' ; m_i the Schur index of the absolutely irreducible component of the representation Γ_i' relative to K' . If the direct product of the representations Γ_i' and Γ_j' contains Γ_k' for $\alpha_{ij}^{(k)}$ times, then the ratio $\frac{\alpha_{ij}^{(k)} m_k}{m_i m_j}$ is an integer.

Let us consider other facts of the theory of representations on an arbitrary field, which can be obtained on the basis of the properties of Φ-characters.

A direct corollary of THEOREM 3.1 and formula (21.1) is the theorem:

The number of irreducible K'-characters of a finite group G sustaining the S-transformation $g \to g^\nu$ (group automorphism) equals the number of K'-divisions of the group sustaining this transformation (this automorphism).

This result also admits such a formulation:

THEOREM 6.1. The number of irreducible representations of a finite group G on a field K', whose characteristic does not divide the order of G, where the representations remain mutually equivalent for a given automorphism $\varepsilon \to \varepsilon^\nu$ of a field of characters (for a given group automorphism), equals the number of K'-divisions of a group which sustain the transformation $g \to g^\nu$ (this automorphism).

For the case of representations on an algebraically closed field, another solution of the question of the number of irreducible representations which transform into equivalent representations for a given group automorphism is given in [11].

Using THEOREM 4.1, the question of the isomorphism of centers of group algebras of certain classes of groups can be solved.

Let us consider the decomposition of the center $Z_{K'}$ of the algebra $R(G,K')$ into a direct sum of subfields of the field $K'(\varepsilon)$, corresponding to the two-sided decomposition (1.1) of the algebra $R(G,K')$:

(24.1) $$Z_{K'} = Z'_1 + \ldots + Z'_q$$

(Z'_i is the center of the ideal I'_i; $i = 1,\ldots,q$).

The minimum idempotents $e'_i \in I'_i \cap Z_{K'}$ are units of the field Z'_i $(i = 1,\ldots,q)$.

It is easy to see that the degree of the field Z'_i on K' equals r_i, the number of minimum idempotents of the center $R(G,\hat{K})$, into the sum of which the e'_i is decomposed [see (20.1)].

Actually, in the extension of the field K' to the field $\hat{K}$, the two-sided ideal $I'_i \subseteq R(G,K')$ transforms into the two-sided ideal $\hat{I}_i = R(G,\hat{K})e'_i$ of the algebra $R(G,\hat{K})$ and the center Z'_i of the ideal I'_i into the center $\hat{Z}_i$ of the ideal $\hat{I}_i$ $(i = 1,\ldots,q)$. The dimensionality of Z'_i on K'

equals the dimensionality of $\hat{Z}_i$ on $\hat{K}$. But the dimensionality of $\hat{Z}_i$ on $\hat{K}$ equals r_i $(i = 1,\dots,q)$. This means that the degree of Z_i' on K' equals r_i.

Let us note that r_i is the number of absolutely irreducible characters in a K'-division of characters X_i, the corresponding K'-division of the idempotents of the center $\{\hat{e}_{i1},\dots,\hat{e}_{ir_i}\}$.

Now, let us assume that the Galois group of the field $K'(\varepsilon)$ on K' is cyclic. Then, by virtue of THEOREM 4.1, a K'-division T_i of the group G corresponds to each K'-division of the characters X_i so that T_i contains as many classes of conjugate elements as there are classes in X_i $(i = 1,\dots,q)$.

Therefore, the numbers $r_1,\dots,r_q$ coincide with the numbers of classes in the corresponding K'-divisions of the group G.

Hence, there at once results

THEOREM 7.1. Let G and H be finite groups; m the least common multiple of the orders of elements of G and H; K' an arbitrary field whose characteristic does not divide m; ε a primitive root of degree m of unity. If the Galois group of the field $K(\varepsilon')$ on K' is cyclic, then the centers of the group algebras $R(G,K')$ and $R(H,K')$ are isomorphic if and only if there exists a mutual one-to-one correspondence between the K' divisions of the groups G and H for which the appropriate K'-divisions of these groups contain the identical number of classes of conjugate elements.

Actually, the fields into a sum of which the centers of the algebras $R(G,K')$ and $R(H,K')$ are decomposed are subfields of the field $K'(\varepsilon)$ and the intermediate subfield of a cyclic extension of the field K' is determined uniquely by its degree on K'.

COROLLARY: If one of the following conditions is satisfied:

a) The order of each of the groups G and H equals p^α or $2p^\alpha$ (p an odd prime),

b) K' is a field of characteristics $p > 0$,

c) K' is a field of real numbers,

then a necessary and sufficient condition of isomorphism of the centers of the group algebras $R(G,K')$ and $R(H,K')$ is the existence of a mutually one-to-one correspondence between the K'-divisions of G and H for which the corresponding K'-divisions of these groups contain the same number of classes.

Actually, in cases a), b), c) , the Galois group of the field $K'(\varepsilon)$ on K' is cyclic (it is assumed, as before, that the characteristic of the field K' does not divide the orders of the groups under consideration).

If the orders of the groups G and H equal 2^m , then the isomorphism of the centers of the group algebras $R(G,K')$ and $R(H,K')$ does not result from the coincidence of the numbers of classes in the corresponding K' divisions of these groups.*

For example, let us consider the centers of group algebras on the field R of rational numbers of two groups of 16-th order:

$$G: \qquad a^8 = 1, \quad b^2 = 1, \quad b^{-1}ab = a^7;$$
$$H: \qquad c^8 = 1, \quad d^2 = 1, \quad d^{-1}cd = c^3$$

Let Z_1 be the center of $R(G,R)$, Z_2 the center of $R(H,R)$. It is easy to obtain the following decompositions for Z_1 and Z_2 :

$$Z_1 \cong R + R + R + R + R + R(\sqrt{2})$$
$$Z_2 \cong R + R + R + R + R + R(\sqrt{-2})$$

This means the algebras Z_1 and Z_2 are not isomorphic. Moreover, the corresponding divisions of these groups contain the same number of elements (R -divisions of the arbitrary finite group coincide with divisions of the group).

The COROLLARY to THEOREM 7.1 remains valid for Abelian groups of order 2^m . This results from the following theorem:

THEOREM 8.1. If G is a finite Abelian group of order h and K'' is a field whose characteristic does not divide h , then in the decomposition of $R(G,K')$ into a direct sum of minimum ideals

$$R(G,K') = I_1' \dotplus \ldots \dotplus I_q'$$

each ideal I_i' is isomorphic to the field $K'(\xi_i)$, where ξ_i is a root of degree α_i ($h \equiv 0 (\mathrm{mod}\ \alpha_i)$) of unity, and the degree of the fields I_i' ($i = 1,\ldots,q$) on K' coincide with the orders of the corresponding K' divisions of G .

* Remark during proof-reading. If G is a group all of whose representations are monomials (M -group), then the degrees of the absolutely irreducible representations of the group coincide with the indices of some of its subgroups. For M -groups, the necessary and sufficient conditions of isomorphism of completely reducible group algebras are included in the coincidence of the indices of the corresponding subgroups. Conditions of the isomorphism of centers of group algebras of an M -group can be obtained by this means and for certain classes of M -groups (for example, for p groups and groups with order without squares) necessary and sufficient conditions of isomorphism of group algebras on an arbitrary field whose characteristic does not divide the orders of the groups considered can be proved. The author published these results partially in [19].

PROOF. Let us denote the idempotent generating the ideal I'_t by e'_i $(i = 1,\ldots,q)$. Identifying elements of the form $\lambda e'_i$ $(\lambda \in K')$ with the elements λ, it can be stated that each field I'_i is obtained by adjoining a finite number of roots of unity of the form ge'_i $(g \in G)$, whose degrees divide h, to the field K'. This means $I'_i = K'(\xi_i)$, where ξ_i is a certain root of unity of order α_i $(h \equiv 0(\mathrm{mod}\ \alpha_i))$ $(i = 1,\ldots,q)$.

Let G be decomposed into a direct product of cyclic groups of orders $h_1,\ldots,h_r$ with the generating elements $a_1,\ldots,a_r$, respectively. Then the complete system of minimum idempotents of $R(G,\hat{K})$ is given by the formulas:

$$e_{i_1,\ldots,i_r} = \frac{1}{h}\sum_{t_1=0}^{h_1-1}\cdots\sum_{t_r=0}^{h_r-1} \varepsilon_1^{-i_1t_1}\cdots\varepsilon_r^{-i_rt_r} a_1^{t_1}\cdots a_r^{t_r} \tag{24'.1}$$

$$(i_j = 0,\ldots,h_j - 1\ ;\ j = 1,\ldots,r)$$

(ε_j is a primitive root of degree h_j of 1 ; $j = 1,\ldots,r$) and the elements of G are expressed as follows in terms of the minimum idempotents of $R(G,\hat{K})$:

$$a_1^{i_1}\cdots a_r^{i_r} = \sum_{t_1=0}^{h_1-1}\cdots\sum_{t_r=0}^{h_r-1} \varepsilon_1^{i_1t_1}\cdots\varepsilon_r^{i_rt_r}\, \hat{e}_{t_1,\ldots,t_r} \tag{24''.1}$$

$$(i_j = 0,\ldots,h_j-1\ ;\ j = 1,\ldots,r)$$

If the element $a^{i_1}\ldots a^{i_r}$ is substituted in conformance with the idempotent $\hat{e}_{i_1,\ldots,i_r}$, then it is easy to see that the number of minimum idempotents of $R(G,\hat{K})$ which are K'-conjugate to $\hat{e}_{i_1,\ldots,i_r}$ equals the number of elements of G which are K'-conjugate to $a_1^{i_1}\ldots a_r^{i_r}$.

Therefore, the orders of K'-divisions of G coincide with the degrees of the fields I'_i on K'.

The THEOREM is proved.

COROLLARY. The group algebras $R(G,K')$ and $R(H,K')$ of the primary Abelian, finite groups G and H are isomorphic if and only if G and H contain the same number of K'-divisions and the orders of the corresponding K'-divisions of these groups coincide.

PROOF. The necessary and sufficient condition of the isomorphism of two fields $K'(\varepsilon_1)$ and $K'(\varepsilon_2)$, where ε_1 and ε_2 are, respectively, roots of degrees p^t and p^m of unity (p is an arbitrary prime), is the coincidence of their degrees on K' since one of them is always isomorphic to the subfield of the other.

The necessary condition of the isomorphism of centers of group algebras of arbitrary groups on an arbitrary field gives the following:

THEOREM 9.1. Let G and H be finite groups; m the least common multiple of the orders of elements of G and H; K' an arbitrary field with characteristic which does not divide m; ξ a primitive root of degree m of unity. Let $K' = P_0 \subset P_1 \subset \ldots \subset P_r = K'(\xi)$ be an increasing chain of fields, where P_{i+1} is the cyclic extension of the field P_i $(i = 0,\ldots,r-1)$. If the centers of the group algebras $R(G,K')$ and $R(H,K')$ are isomorphic, then between the P_i -divisions of the groups G and H there exists a mutually one-to-one correspondence for which the corresponding P_i -divisions of these groups contain the same number of P_{i+1} -divisions $(i = 0,\ldots,r-1)$.

PROOF. Let F_i be a Galois group of the field $K'(\xi)$ on P_i; F_{i+1} a Galois group of the field $K'(\xi)$ on P_{i+1}. Then F_i/F_{i+1} is a Galois group of the field P_{i+1} on P_i. By virtue of the conditions of the theorem, this group is cyclic.

Let $T_1,\ldots,T_m$ be P_{i+1} -divisions of G; $e_1,\ldots,e_m$ minimum idempotents of the center of the group algebra $R(G,P_{i+1})$. The groups Φ_i and Φ_{i+1} of the S -mapping $g \to g^{\nu}$ $(g \in G)$ correspond to the groups F_i and F_{i+1} to which they are isomorphic, where Φ_i/Φ_{i+1} is a cyclic group since $\Phi_i/\Phi_{i+1} = F_i/F_{i+1}$.

Evidently, the mappings $\varphi \in \Phi_{i+1}$ retain each idempotent e_j and each P_{i+1} -division T_j $(j = 1,\ldots,m)$ in place. Hence, there results that the transformations $\varphi \in \Phi_i$ from one adjacent class Φ_i to Φ_{i+1} of the same affect the idempotents e_j and the P_{i+1} -divisions T_j $(j = 1,\ldots,m)$. This means that the factor-group Φ_i/Φ_{i+1} can be considered as a cyclic group of $S\text{-}I_{i+1}$ -transformations of the group G (see DEFINITION 5.1).

On the basis of THEOREM 4.1, the set $\{T_1,\ldots,T_m\}$ of P_{i+1} -divisions of G and the set $\{e_1,\ldots,e_m\}$ of minimum idempotents of the center $R(G,P_{i+1})$ decompose under the effect of the group Φ_i/Φ_{i+1} of $S\text{-}\Phi_{i+1}$ transformations into the same number of transitivity regions so that the corresponding transitivity regions of these sets contain the same number of elements (P_{i+1} -divisions and idempotents e_j, respectively):

$$\{T_1,\ldots,T_m\} = M_1 \cup \ldots \cup M_{k_i}\ ; \quad M_i = \{T_{i1},\ldots,T_{is_1}\}$$

$$\{e_1,\ldots,e_m\} = E_1 \cup \ldots \cup E_{k_i}\ ; \quad E_i = \{e_{i1},\ldots,e_{is_1}\}$$

The number of P_i -divisions of the group G equals k and coincides with the number of minimum idempotents of the center $R(G,P_i)$. Each P_i division $\tilde{T}_j$ $(j = 1,\ldots,k_i)$ is a set-theoretic sum of elements of the group G belonging to the P_{i+1} -divisions $T_{j1},\ldots,T_{js_j}$ of the set M_j $(j = 1,\ldots,k_i)$. The minimum idempotents $\tilde{e}_1,\ldots,\tilde{e}_{k_i}$ of the center $R(G,P_i)$ are given by the formulas:

$$\tilde{e}_j = e_{j1} + \ldots + e_{js_j} \qquad (j = 1,\ldots,k_i)$$

If the centers of the group algebras $R(G,K')$ and $R(H,K')$ are isomorphic, then the centers of the group algebras $R(G,P_j)$ and $R(H,P_j)$ $(j = 0,\ldots,r)$ are also isomorphic.

This means that the group algebra $R(H,P_i)$ contains exactly k_i minimum idempotents of the center $\tilde{e}'_1,\ldots,\tilde{e}'_{k_i}$ and these latter are decomposed in $R(H,P_{i+1})$ into the same number of minimum idempotents of the center $R(H,P_{i+1})$ as are the corresponding idempotents $\tilde{e}_1,\ldots,\tilde{e}_{k_i}$:

$$\tilde{e}'_j = e'_{j1} + \ldots + e'_{js_j} \qquad (j = 1,\ldots,k_i)$$

Reasoning analogously for H as was done for G, we obtain that the numbers $s_1,\ldots,s_{k_i}$ coincide with the numbers of P_{i+1} -divisions in the P_i -divisions of H $(i = 0,\ldots,r-1)$.

The THEOREM is proved.

§ 2. INDUCED REPRESENTATIONS

1. Relations between Φ -Characters of Groups and Subgroups. In this paragraph, we assume that the algebraically clsoed field $\hat{K}$ has characteristic zero.

Let G be a group of order h ; H a subgroup of order h' of the group G ; Φ such a group of S -mappings of the group G that each S mapping $\varphi \in \Phi$ transforms H into itself and is a S -mapping of this subgroup. Then Φ induces the group Φ' of S -mappings of the group H.

Each representation Γ of the group G on the field $\hat{K}$ induces a representation of the subgroup H which we will denote by $\Gamma\downarrow(H)$.

Conversely, the representation Γ' of the subgroup H induces the representation $\Gamma'\uparrow(G)$ of the group G.

If $\chi(a)$ is a character of the representation Γ' of the subgroup H then the character $\chi^*(g)$ of the induced representation of the group G is given by the formula:

$$\chi^*(g) = \frac{1}{h'} \sum_{c \in G} \chi(c^{-1}gc) \qquad (\chi(c^{-1}gc) = 0 \text{ if } c^{-1}gc \bar{\in} H) \tag{1.2}$$

Formula (1.2) can also be written in the following form:

$$\chi^*(g) = \sum_{a \in C_g \cap H} \frac{h}{h' \cdot h_g} \chi(a) \tag{1'.2}$$

where C_g is the class of conjugate elements of the group G containing the element g ; h_g is the order of the class C_g .

If $\varphi \in \Phi$, the following formula holds:

$$C_{\varphi(g)} \cap H = \varphi(C_g \cap H) \tag{2.2}$$

Actually, if $x \in C_g \cap H$, then $\varphi(x) \in H$ (H sustains the transformation from the group Φ according to the condition) and $\varphi(x) \in C_{\varphi(g)}$ because φ is an S -mapping of the group G .

Conversely, if $x \in C_{\varphi(g)} \cap H$, then there exists an element $y \in C_g$ such that $\varphi(y) = x$ but since $x \in H$, then also $y \in H$, i.e., $y \in C_g \cap H$.

LEMMA 1.2. If Γ is a Φ -representation of G , then $\Gamma\downarrow(H)$ is a Φ' -representation of H . If Γ' is a Φ' -representation of H , then $\Gamma'\uparrow(G)$ is a Φ -representation of G (see DEFINITION 4.1).

PROOF. The first statement of the lemma is obvious. Let $\chi(a)$ be a character of the Φ' -representation Γ' of the group H . By virtue of (1'.2), the character $\chi^*(g)$ of the representation $\Gamma'\uparrow(G)$ is expressed by the formula:

$$\chi^*(g) = \sum_{a \in C_g \cap H} \frac{h}{h' \cdot h_g} \chi(a)$$

If $\varphi \in \Phi$ then because of (2.2)

$$\chi^*(\varphi(g)) = \sum_{a \in C_{\varphi(g)} \cap H} \frac{h}{h' \cdot h_g} \cdot \chi(a) = \sum_{a \in \varphi(C_g \cap H)} \frac{h}{h' \cdot h_g} \chi(a)$$
$$= \sum_{a \in C_g \cap H} \frac{h}{h' \cdot h_g} \chi(\varphi(a)) = \sum_{a \in C_g \cap H} \frac{h}{h' \cdot h_g} \chi(a) = \chi^*(g)$$

This means $\Gamma'\uparrow(G)$ is a Φ -representation of G.

The LEMMA is proved.

Let $\Gamma_1, \ldots, \Gamma_t$ be irreducible Φ -representations of the group G ; χ_i the character of the representation Γ_i $(i = 1, \ldots, t)$; $\Gamma_1', \ldots, \Gamma_q'$ the Φ' -irreducible representations of the subgroup H ; χ_i' the character of the representation Γ_i' $(i = 1, \ldots, q)$; r_i (r_i') the number of characters of a Φ -division of characters of the group G (of a Φ' -division of the characters of the group H , respectively) corresponding to the Φ-character χ_i (χ_i') .

Let us consider the representation $\Gamma_i\downarrow(H)$ of the subgroup H induced by the representation Γ_i of the group G .

By virtue of LEMMA 1.2, $\Gamma_i\downarrow(H)$ is a Φ'-representation of the group H . This means $\Gamma_i\downarrow(H)$ decomposes into a sum of Φ'-irreducible representations (see LEMMA 4.1):

$$\Gamma_i\downarrow(H) = \lambda_{i1}\Gamma_1' + \dots + \lambda_{iq}\Gamma_q' \tag{3.2}$$

(λ_{ij} are non-negative integers).

In conformance with (3.2), we obtain the formula

$$\chi_i(a^{-1}) = \sum_{j=1}^{q} \lambda_{ij}\chi_j'(a^{-1}) \qquad (a \in H) \tag{4.2}$$

Let us multiply both sides of (4.2) by $\chi_s'(a)$ and let us sum over all elements $a \in H$ taking (IV'') (§ 1) into account:

$$\begin{aligned} \sum_{a\in H} \chi_s'(a)\chi_i(a^{-1}) &= \sum_{a\in H}\sum_{j=1}^{q} \lambda_{ij}\chi_j'(a^{-1})\chi_s'(a) \\ &= \sum_{j=1}^{q} \lambda_{ij}\Bigl(\sum_{a\in H} \chi_s'(a)\chi_j'(a^{-1})\Bigr) = \lambda_{is}r_s'h' \end{aligned} \tag{5.2}$$

Multiplying the right and left sides of (5.2) by $\dfrac{\chi_i(g)}{r_i}$ $(g \in G)$ and summing over i , we obtain the relation

$$\sum_{a\in H} \chi_s'(a) \sum_{i=1}^{t} \chi_i(g)\chi_i(a^{-1}) = \sum_{i=1}^{t} \frac{1}{r_i}\lambda_{is}r_s'h'\cdot\chi_i(g) \tag{6.2}$$

Let T_g be a Φ-division of the group G containing the element g , l_g the order of T_g . By virtue of (III'), the following equality follows from (6.2):

$$\frac{1}{h'}\sum_{a\in T_g\cap H} \frac{h}{l_g}\chi_s'(a) = \sum_{i=1}^{t} \frac{r_s'}{r_i}\lambda_{is}\,\chi_i(g) \tag{7.2}$$

Let the Φ-division T_g consist of the classes $C_g, C_{\varphi_1(g)}, \dots, C_{\varphi_{k-1}(g)}$ ($\varphi_i \in \Phi$; $i = 1,\dots,k-1$). Then $l_g = kh_g$, where h_g is the order of the class C_g . Since the character $\chi_s'(a)$ is a function of the Φ'-division of the group H (see THEOREM 2.1), then because of (2.2)

$$\frac{1}{h'}\sum_{a\in T_g\cap H}\frac{h}{l_g}\chi_s'(a) = \frac{k}{h'}\sum_{a\in C_g\cap H}\frac{h}{k\cdot h_g}\chi_s'(a) = \frac{1}{h'}\sum_{a\in C_g\cap H}\frac{h}{h_g}\chi_s'(a) \tag{8.2}$$

By virtue of (8.2), the equality (7.2) can be written as:

$$\frac{1}{h'} \sum_{a \in C_g \cap H} \frac{h}{h_g} \chi_s'(a) = \sum_{i=1}^{t} \lambda_{is} \frac{r_s'}{r_i} \chi_i(g) \tag{9.2}$$

The left side of (9.2) is a character of the Φ-representation $\Gamma_s'\uparrow(G)$ of the induced Φ'-representation Γ_s' of the group H [see (1'.2)].

Hence, from (9.2) there results the statement generalizing the Frobenius duality theorem [2]:

THEOREM 1.2. Let G be a group; H a subgroup of G; Φ a group of S-mappings of G which induces a group of S-mappings of H; Γ and Γ' are Φ-irreducible representations of the group G and the subgroup H, respectively. If the representation $\Gamma\downarrow(H)$ contains Γ' α times, then the Φ-representation $\Gamma'\uparrow(G)$ contains Γ $\alpha \frac{r'}{r}$ times, where r (r') is the number of absolutely irreducible representations into the sum of which Γ (Γ' correspondingly) is decomposed.

The following holds for induced representations over an arbitrary field

THEOREM 2.2. Let Γ and Γ' be irreducible representations of the group G and its subgroup H, respectively, over an arbitrary field K' of characteristic zero. Let the minimum, two-sided ideal I (I'), isomorphic to the complete matrix ring on a field D (D') correspond to the representation Γ (Γ') in $R(G,K')$ $(R(H,K'))$.

If the representation $\Gamma\downarrow(H)$ contains the representation Γ' α times, then the representation $\Gamma'\uparrow(G)$ contains the representation Γ $\alpha \frac{d'}{d}$ times, where d (d') is the dimensionality of the field D (D') over K'.

The proof of the theorem is based on formula (IV''') and (III'') of § 1 by the same means as the proof of THEOREM 1.2 is based on (IV'') and (III').

Relations (IV''') and (III''), respectively, are obtained from (IV'') and (III') by replacing r_i by $r_i m_i^2 = d_i$ (r_i is the dimensionality of the center of the field D_i over K'; m_i^2 is the dimensionality of D_i over its center). Consequently, the ratio $\frac{d'}{d}$ appears instead of the ratio $\frac{r'}{r}$ in the formulation of THEOREM 2.2. THEOREM 2.2. is also a generalization of the Frobenius duality theorem.

2. $\mathfrak{p}$-adic Ring of K'-Characters. The Brauer [5] theorem plays an important part in the theory of induced representations of a group:

Each character of a finite group G is an integer linear combination of characters induced by characters of elementary subgroups of the group G.

The direct product of a p-group and a cyclic group whose order is not divided by p is called an elementary group.

The Brauer theorem is obtained by Roquette [7] as a consequence of a number of structural theorems relative to a $\mathfrak{p}$-adic ring of absolutely irreducible characters of a finite group.

A $\mathfrak{p}$-adic ring of characters of irreducible representations of a group G over an arbitrary field K' of zero characteristic is investigated in this paragraph by the Roquette method. A generalization of the Brauer theorem for representations over the field K' is obtained as a consequence. Other results of [6], [13], related to the Brauer theorem, are also generalized.

As before, let us use the notations (A).

DEFINITION 1.2. Let H be a normal divisor of the group G. The group of inner automorphisms of the group G induces a group Φ of S-mappings of H. Let us agree to call Φ-conjugate elements of H, Φ-conjugate characters of H and Φ-conjugate minimum idempotents of the center $R(H,\hat{K})$ G-conjugates.

Let us call Φ-divisions of the group H, characters of H and minimum idempotents of the center $R(H,\hat{K})$ G-divisions.

Evidently, G-divisions of the group H are the class of conjugate elements of the group G contained in H.

LEMMA 2.2. Let us assume that H is a subgroup of index m of the group G. If the idempotent $e \in R(H,K')$ generates a left ideal I of dimensionality r over K' in $R(H,K')$, then the dimensionality of the left ideal $\tilde{I} = R(G,K')\cdot e$ over K' equals mr.

PROOF. Let $a_1e, \ldots, a_re$ (a_i H; $i = 1,\ldots,r$) be a basis of I over K'; $b_1,\ldots,b_m$ a system of representations of left neighboring classes of the group G on H. Then it is easy to verify that the elements b_ja_ie ($i = 1,\ldots,r$; $j = 1,\ldots,m$) form a basis $\tilde{I}$ over K'.

THEOREM 3.2. Let H be a normal divisor of index m of the the group G ; e the idempotent of the center $R(H,K')$ generating the minimum two-sided ideal I , isomorphic to the complete matrix ring of order r over the field D , in $R(H,K')$.

If the number of minimum idempotents of the center $R(H,K')$, which are G -conjugate to e , equals m , then the sum of these idempotents $\tilde{e} = e_1 + \dots + e_m$ $(e_1 = e)$ is a minimum idempotent of the center $R(G,K')$ and the ideal $\tilde{I} = R(G,K')\tilde{e}$ is isomorphic to the complete matrix ring of order mr over the same field D .

PROOF. Let $b_1,\dots,b_m$ $(b_1 = 1)$ be a system of representations of neighboring classes of the group G on H ; $I_i = R(H,K')\cdot e_i$ $(i = 1,\dots,m)$. Changing, perhaps, the numbering of the elements b_i , it can be considered, by virtue of the conditions of the theorem, that

(10.2) $$b_i^{-1} I_1 b_i = I_i \quad (i = 1,\dots,m)$$

The ideals $I_1,\dots,I_m$ are orthogonal in pairs:

(11.2) $$I_i I_j = 0 , \quad \text{if } i \neq j \quad (\text{see } (1.1))$$

Let $u \in I_1$ be a minimum idempotent of $R(H,K')$

$$U = R(H,K')u \; ; \qquad \tilde{U} = R(G,K')u$$

If $a_1 u,\dots,a_q u$ $(a_i \in H \; ; \; i = 1,\dots,a \; ; \; a_1 = 1)$ is the basis of the minimum ideal U over K' , then the elements $b_j a_i u$ $(j = 1,\dots,m;$ $i = 1,\dots,q)$ form a basis of the ideal $\tilde{U}$ over K' (see LEMMA 2.2). Each element $x \in \tilde{U}$ is represented uniquely as:

(12.2) $$x = \sum_{i,j} \lambda_{ij} b_j a_i u \qquad (\lambda_{ij} \in K' \; ; \; i = 1,\dots,q; \; j = 1,\dots,m)$$

Let $\tilde{D}$ be a ring of operator endomorphisms of an additive group of the ideal $\tilde{U}$ (the elements of $R(G,K')$ are considered as left ideals). The arbitrary endomorphism $\tilde{\theta} \in \tilde{D}$ is given by the formula

(13.2) $$\tilde{\theta}_c(x) = xc = xuc \qquad (x, \; c \in \tilde{U})$$

(u is the right unit ideal $\tilde{U}$) and formula (13.2) yields the operator endomorphism $\tilde{\theta}_c \in \tilde{D}$ for any element $c \in \tilde{U}$. The endomorphisms $\tilde{\theta}, \tilde{\eta} \in D$ are equal if and only if $\tilde{\theta}(u) = \tilde{\eta}(u)$.

If the endomorphisms $\tilde{\theta}_{\lambda u}$ $(\lambda \in K')$ are identified with the corresponding elements λ of the field K' , then it can be stated, by virtue of (12.2) and (13.2), that each endomorphism $\tilde{\theta} \in \tilde{D}$ is represented as:

(13'.2) $$\tilde{\theta} = \sum_{i,j} \lambda_{ij} \tilde{\theta}_{ij} \qquad (\lambda_{ij} \in K' \; ; \; i = 1,\dots,q \; ; \; j = 1,\dots,m)$$

where

(14.2) $\qquad \tilde{\theta}_{ij}(x) = xb_j a_i u = xub_j a_i u \qquad (x \in \tilde{U}\ ;\ i = 1,\dots,q;\ j = 1,\dots,m)$
We have $\tilde{\theta}_{ij}(x) = xub_j a_i u = xb_j(b_j^{-1}ub_j)a_i u = 0$ for $j > 1$ because $b_j^{-1}ub_j \in I_j$ by virtue of (10.2) and $(b_j^{-1}ub_j)(a_i u) = 0$ $(a_i u \in I_1)$ as a consequence of (11.2).

Therefore, by virtue of (13.2) and (13'.2), we obtain for an arbitrary endomorphism $\tilde{\theta} \in \tilde{D}$:

(15.2) $\qquad \tilde{\theta}(x) = x(\lambda_1 a_1 + \dots + \lambda_q a_q)u \qquad (x \in \tilde{U}\ ;\ \lambda_i \in K'\ ;\ i = 1,\dots,q)$

Since u is a minimum idempotent of $R(H,K')$, then the ring D' of operator endomorphisms of the ideal U is a field. D' is inversely isomorphic to the field D because $u \in I_1$.

The operator endomorphism $\theta \in D'$ is expressed by the formula

(15'.2) $\quad \theta(x) = x(\lambda_1 a_1 + \dots + \lambda_q a_q)u \qquad (x \in U\ ;\ \lambda_i \in K'\ ;\ i = 1,\dots,q)$

The correspondence $\tilde{\theta} \to \theta$ ($\tilde{\theta}$ is given by (15.2) and θ by (15'.2)) is an isomorphism of the mapping of D on D' .

Actually, $U \subseteq \tilde{U}$ and $\tilde{\theta}$ is the continuation of the mapping θ from U onto $\tilde{U}$, where $\tilde{\theta}$ is defined uniquely by the endomorphism θ since $\tilde{\theta}(u) = \theta(u)$. This means the ring $\tilde{D}$ is a field from which there results that u is a minimum idempotent of $R(G,K')$. Since $\tilde{D} \cong D'$ and D' is inversely isomorphic to D then $\tilde{D}$ is inversely isomorphic to D .

Let the dimensionality of the field D over K' equal s . Since I_1 is the complete matrix ring of order r over D , then the dimensionality of I_1 over K' is $r^2 s$, the dimensionality of the ideal $R(H,K')\cdot\tilde{e}$ is $mr^2 s$ by virtue of (10.2) and the dimensionality of the ideal $\tilde{I}$ over K' is $m^2 r^2 s$ as a consequence of LEMMA 2.2.

U is a minimum left ideal of a two-sided minimum ideal $I_1 \subset R(H,K')$. This means that the dimensionality of U over K' is $\frac{r^2 s}{r} = rs$ and the dimensionality of $\tilde{U}$ over K' is rsm on the basis of LEMMA 2.2.

Let $I' \subseteq \tilde{I}$ be a minimum two-sided ideal of $R(G,K')$ containing a minimum left ideal $\tilde{U}$. Then I' is the complete matrix ring of order t over a field inversely isomorphic to the field $\tilde{D}$ of operator endomorphisms of the ideal $\tilde{U}$, i.e., over the field D . Therefore, the dimensionality of I' over K' is st^2 and the dimensionality of $\tilde{U}$ over K' is $\frac{st^2}{t} = st$. Hence, $rsm = st$ from which $t = rm$.

Thus the dimensionality of I' over K' is $s(rm)^2$ and since the dimensionality of $\tilde{I}$ over K' is sr^2m^2 also, then $I' = \tilde{I}$.

We have shown that the ideal I is a complete matrix ring of order rm on a field D.

The THEOREM is proved.

Let G be a finite group; n the least common multiple of the orders of the elements of G; K' an arbitrary field of characteristic zero; ε a primitive n-th root of unity.

DEFINITION 2.2. We shall call the set $N_{K'}(a)$ of all elements of $g \in G$, such that $g^{-1}ag = a^{\mu}$ a K'-normalizer of the element $a \in G$, where $\varepsilon \to \varepsilon^{\mu}$ is an automorphism of the Galois group of the field $K'(\varepsilon)$ over K'.

Evidently, the K'-normalizer of an arbitrary element of the group G is a subgroup of this group.

DEFINITION 3.2. We call the group G with the following properties a K'-elementary group:

1) G is a semi-direct product: $G = H \cdot F$, where the normal divisor $H = (a)$ is a cyclic group whose order is not divided by the prime p and F is a p-group.

2) The K'-normalizer of the element a in G coincides with G.

LEMMA 3.2. Let G be the semi-direct product of a cyclic group H whose order is mutually prime to p and the p-group F (H is a normal divisor of G); $H' = (b)$ is the primary component of the group H; N the normalizer of the element b in G. Then N is also a normalizer of any element of the subgroup H' different from unity.

PROOF. Let us assume that the order b is p_1^{α} (p_1 is a prime; $p \neq p_1$).

Let us take an arbitrary element $a \in F$ satisfying the condition $a \notin N \cap F$, and an element b^s H' ($s \not\equiv 0 (\mathrm{mod}\ p_1^{\alpha})$). The LEMMA will evidently be proved if it isestablished that $a^{-1}b^s a \neq b^s$.

Let us assume the reverse. Let $a^{-1}b^s a = b^s$. If $a^{-1}ba = b^{\mu}$, then $a^{-1}b^s a = b^{\mu s}$ and, therefore, $\mu s = s \ (\mathrm{mod}\ p_1^{\alpha})$. This means

$$(16.2) \qquad \mu \equiv 1 \left(\mathrm{mod}\ \frac{p_1^{\alpha}}{d}\right), \text{ where } d = (p_1^{\alpha}, s)$$

Evidently $d > 1$, since otherwise

$$a^{-1}ba = b, \text{ i.e., } a \in N$$

Let $\frac{p_1^{\alpha}}{d} = p_1^{j}$ $(0 < j < \alpha)$. By virtue of (16.2), $\mu = 1 + p_1^{j}k$, from which

$$\mu^{p_1^{\alpha-j}} = (1 + p_1^j k)^{p_1^{\alpha-j}} \equiv 1 \pmod{p_1^\alpha})$$

Therefore μ belongs to the exponent p_1^t $(t > 0)$ modulo p_1^α .

If the order a equals p^ν $(\nu > 0)$, then from the equality $a^{-1}ba = b^\mu$ there results the relation $a^{p^\nu} b a^{p^\nu} = b^{\mu p^\nu} = b$ with the result that

$$\mu^{p^\nu} \equiv 1 \pmod{p_1^\alpha})$$

This means the exponent p_1^t $(t > 0)$, to which μ belongs modulo p_1^α , divides p^ν $(\nu > 0)$.

The contradiction obtained shows that $a^{-1}b^s a \neq b^s$.

COROLLARY. Each G -division of the group H' which differs from unity contains $(G:N)$ elements and an arbitrary G -division of the characters of the group H' which does not contain the principal character $\chi(g) = 1$ consists of $(G:N)$ characters.

PROOF. A statement relative to the G -divisions of H' results directly from LEMMA 3.2.

If $a \bar{\in} N$ $(a \in F)$, then the inequality $\chi(a^{-1}ga) \neq \chi(g)$ $(g \in H')$ is satisfied for any character $\chi(g) \equiv 1$ of the group H' .

Actually, if a certain character $\chi(g) \neq 1$ were to sustain the automorphism $a^{-1}ga$, then on the basis of THEOREM 3.1, an element g H' $(g \neq 1)$ would be found such that $a^{-1}ga = g$ and this contradicts the LEMMA 3.2 . This means N is a subgroup which retains any character $\chi(g) \neq 1$ in place, from which it follows that the number of characters G -conjugate to $\chi(g)$ is $(G:N)$.

LEMMA 4.2. Let $G = H \cdot F$ be a K' -elementary group (H is a cyclic group and F a p-group); $H' = (b)$ is the primary component of order $h' = p_1^\alpha$ of the group H ; N the normalizer of the element b in G . The subgroup N can be represented as the direct product: $N = H' x Q$, where $Q = G' \cdot N'$, $H = H' x G'$, $N' = N \cap F$.

Let $\chi_0(g), \ldots, \chi_{m-1}(g)$ are constant K' -characters of the group N in adjacent classes with respect to Q , which are obtained by a natural continuation of the irreducible K' -characters $\chi_0(a), \chi_1(a), \ldots, \chi_{m-1}(a)$ of the group H' ($\chi_0(a)$ is the principal character).

Then the function

$$\chi_i'(g) = \begin{cases} \tilde{\chi}_i(g) & \text{if } g \in N \\ 0 & \text{if } g \bar{\in} N \end{cases} \qquad (i = 1,\ldots,m-1) \tag{16'.2}$$

are irreducible K' -characters of the group G .

PROOF. The subgroups N and Q are normal divisors of the group G . Let $\hat{\chi}_1(a)$ (the principal character), $\hat{\chi}_2(a),\ldots,\hat{\chi}_{h'}(a)$ be absolutely irreducible characters of the group H' ; h the order of G ; $d = (G:N)$.

By virtue of the corollary to LEMMA 3.2, the set $\hat{X} = \{\hat{\chi}_2,\ldots,\hat{\chi}_{h'}\}$ is decomposed into G -divisions each of which contains exactly d characters:

$$\begin{aligned} &\hat{X} = \hat{X}_1 \cup \ldots \cup \hat{X}_q \;; \quad \hat{X}_i \cap \hat{X}_j = \Lambda \quad \text{if } i \neq j \\ &\hat{X}_i = \{\hat{\chi}_{i1},\ldots,\hat{\chi}_{id}\} \qquad (i = 1,\ldots,q) \end{aligned} \tag{16''.2}$$

The set X is also decomposed into K' -divisions of characters:

$$X = T_1 \;\ldots\; T_s$$

According to the definition of a K' -elementary group, the group Φ of S -mappings $a \to a^{\nu}$ of the group H' corresponding to the Galois group of the field $K'(\varepsilon)$ over K' (see DEFINITION 8.1) contains the group Φ' of automorphisms φ_x ($\varphi_x(a) = x^{-1}ax$; $x \in G$; $a \in H'$) of the group H' as a subgroup. Here Φ' is a normal divisor of Φ since Φ is an Abelian group.

Using LEMMA 6.1, we obtain that each K' division of characters T_i of the group H' consists of several G -divisions $\hat{X}_j$:

$$T_i = \{\hat{X}_{i1},\ldots,\hat{X}_{ir_i}\} \qquad (i = 1,\ldots,s \;;\; \sum_{i=1}^{s} r_i = q) \tag{17.2}$$

The following sum corresponds to each G -division of the characters $\hat{X}_i$:

$$\chi''(a) = \hat{\chi}_{i1}(a) + \ldots + \hat{\chi}_{id}(a) \qquad (a \in H' \;;\; i = 1\ldots,q) \tag{18.2}$$

In conformance with (17.2), (18.2) and (21.1) the following formulas hold for the characters $\chi_1(a),\ldots,\chi_{m-1}(a)$ of irreducible representations of the group H' over K' :

$$\chi_i(a) = \chi''_{i1}(a) + \ldots + \chi''_{ir_i}(a) \qquad (i=1,\ldots,s \;;\; s = m-1) \tag{19.2}$$

(see the formulation of LEMMA 4.2).

The minimum idempotent e_i $(i = 1,\ldots,s)$ of the algebra $R(H',K')$ corresponds to the K' -character χ_i of the group H' :

$$e_i = \frac{1}{h'} \sum_{a \in H'} \chi_i(a^{-1})a \qquad (i = 1,\ldots,s) \tag{19'.2}$$

The idempotent e_i generates a minimum ideal V_i in $R(H',K')$, which is a field, the extension of the field K'. Evidently, e_i $(i = 1,\ldots,s)$ is an idempotent of the center $R(G,K')$. It is easy to show that the elements

$$\hat{e}_i = \frac{d}{h} \sum_{a \in H'} \hat{\chi}_i(a^{-1})a \sum_{c \in Q} c \quad (\tfrac{h}{d} \text{ is the order of } N;\ i = 1,\ldots,h')$$

are minimum idempotents of the center of the group algebra $R(N,\hat{K})$ ($\hat{K}$ is the algebraic closure of the field K').

In view of (16''.2), the set $E = e_2,\ldots,e_{h'}$ is separated into G-divisions:

$$\hat{E} = \hat{E}_1 \cup \ldots \cup \hat{E}_q\ ;\quad \hat{E}_i = \{\hat{e}_{i1}, \ldots, \hat{e}_{id}\} \qquad (i = 1,\ldots,q)$$

(the idempotent $\hat{e}_{ij}$ corresponds to the character $\hat{\chi}_{ij}$).

Since N is a normal divisor of index d in G, then on the basis of THEOREM 3.2, we conclude that the idempotents

$$(20.2) \qquad e_i'' = \hat{e}_{i1} + \ldots + \hat{e}_{id} = \frac{d}{h} \sum_{a \in H'} \left(\hat{\chi}_{i1}(a^{-1}) + \ldots + \hat{\chi}_{id}(a^{-1})\right) a \cdot \sum_{c \in Q} c$$
$$= \frac{d}{h} \sum_{a \in H'} \chi_i''(a^{-1}) \cdot \sum_{c \in Q} c \qquad (i = 1,\ldots,q)$$

(see (18.2)) are minimum idempotents of the center $R(G,\hat{K})$, to which correspond the absolutely irreducible representations of the group G of the same degree d.

Because of (17.2), (18.2) and (19.2), the set E'' of minimum idempotents of the center $R(G,\hat{K})$: $E'' = \{e_1'',\ldots,e_q''\}$, decomposes into the K'-divisions: $E'' = E_1'' \cup \ldots \cup E_s''$, $E_i'' = \{e_{i1}'',\ldots,e_{ir_i}''\}$ $(i = 1,\ldots,s;$ $s = m-1)$. Combining idempotents from one K'-division E_i'', we obtain the minimum idempotents of the center $R(G,K')$ according to (20.1):

$$(21.2) \qquad e_i' = e_{i1}'' + \ldots + e_{ir_i}'' = \frac{d}{h} \sum_{a \in H'} (\chi_{i1}''(a^{-1}) + \ldots + \chi_{ir_i}''(a^{-1}))a \cdot \sum_{c \in Q} c$$
$$= \frac{d}{h} \sum_{a \in H'} \chi_i(a^{-1})a \sum_{c \in Q} c = \frac{d}{h} \sum_{g \in N} \tilde{\chi}_i(g^{-1})g = \frac{d}{h} \sum_{g \in G} \chi_i'(g^{-1})g$$
$$(i = 1,\ldots,s\ ;\ s = m-1)$$

Let Γ_i' be an irreducible representation of the group G over K' corresponding to the idempotent e_i' $(i = 1,\ldots,s)$; $\tilde{\chi}_i'$ is a character of the representation Γ_i'.

Comparing formula (21.2) with (20.1) and (21.1), we conclude that

$$(21'.2) \qquad \tilde{\chi}_i'(g) = m_i \chi_i'(g) \qquad (i = 1,\ldots,s)$$

where m_i is the Schur index corresponding to the character $\tilde{\chi}_i'(g)$.

Let us be certain that $m_i = 1$ $(i = 1,\dots,s)$. Let $G_1 = G'\cdot F$. Let us put

(22.2) $\tilde{e} = \frac{h'}{h} \sum_{c \in G_1} c$; $u_i = e_i\tilde{e}$ $(i = 1,\dots,m-1)$ ($\frac{h}{h'}$ is the order of G_1)

Evidently $\tilde{e}$ is an idempotent of the algebra $R(G,K')$. Because of the equality $e_i\tilde{e} = \tilde{e}e_i$, u_i is an idempotent of $R(G,K')$. Since $e_ie_i' = e_i$, then $u_ie_i' = u_i$ and, therefore

$$u_i \in I_i' = R(G,K)e_i' \qquad i = 1,\dots,m-1)$$

The dimensionality of the ideal I_i' over the field K' equals r_id^2 (see LEMMA 4 of [18]). Because r_i is the dimensionality of the center of the simple algebra I_i' over the field K' , the dimensionality of the minimum left ideal of the algebra $R(G,K')$ contained in I_i' is r_idm_i and this means that the dimensionality of an arbitrary left ideal $I \subseteq I_i'$ is not less than the number r_idm_i . On the basis of the same LEMMA, the dimensionality of the left ideal $R(G,K)u_i$ over K' equals r_id . Hence, $r_id \geq r_idm_i$, from which $m_i = 1$. Because of (21'.2), the LEMMA is proved.

COROLLARY. If $G = H\cdot F$ is a K' -elementary group (H is a cyclic group, F a p -group), then each irreducible K' -character of the group H is induced by a certain irreducible K' -character of the group G .

PROOF. Let us consider the decomposition of the group H into the direct product of primary cyclic components

$$H = H_1 \times \dots \times H_s$$

Each irreducible K' -character χ of the group H induces an irreducible K' -character $\chi^{(j)}$ of this group $(j = 1,\dots,s)$ on H_j . If χ is the principal character of H , then χ is evidently induced by the principal character of the group G .

Let the characters $\chi^{(1)},\dots,\chi^{(t)}$ satisfy the condition $\chi^{(j)}(a) \neq 1$ $(a \in H_j$; $j = 1,\dots,t$; $t \leq s)$ and let the characters $\chi^{(t+1)},\dots,\chi^{(s)}$ be the principal characters.

Because of LEMMA 4.2, the character $\chi^{(j)}$ $(j = 1,\dots,t)$ is induced by an irreducible K' -character $\chi'^{(j)}$ of the group G , which is expressed by a formula of the form of (16'.2).

Now it is easy to see that the character χ is induced on H by a K' -character $\tilde{\chi}(g) = \chi'^{(1)}(g) \dots \chi'^{(t)}(g)$ of the group G . Since χ

is an irreducible K'-character, then $\tilde{\chi}(g)$ will also be an irreducible K'-character of G.

The corollary to LEMMA 4.2 is used in [12].

The following notations will be required in the sequel:

$$(D)\begin{cases} G & \text{an arbitrary finite group} \\ n & \text{the least common multiple of the orders of the elements } G \\ \varepsilon & \text{a primitive } n\text{-th root of unity} \\ K' & \text{an arbitrary field of characteristic zero} \\ \chi_1',\dots,\chi_r' & \text{irreducible } K'\text{-characters of the group } G \\ C & \text{a ring of rational integers} \\ \Gamma_p' & \text{a field of rational } p\text{-adic numbers} \\ \mathfrak{p} & \text{a simple ideal of the field } \Gamma_p'(\varepsilon) = P \\ I_{\mathfrak{p}} & \text{a ring of } \mathfrak{p}\text{-adic integers of the field } P \end{cases}$$

If $R \supseteq C$ is an arbitrary ring, then a set X_R of all possible linear combinations of irreducible K'-characters $\chi_1',\dots,\chi_r'$ with coefficients from R can be formed:

(21''.2) $\quad X_R = \{\alpha_1\chi_1' + \dots + \alpha_r\chi_r'\} \qquad (\alpha_i \in R\ ;\ i = 1,\dots,r)$

If the operations of addition and multiplication of elements from X_R are defined by considering them as functions prescribed on G, then X_R is transformed into a ring. This results from the first relation between the irreducible K'-characters:

$$\chi_i'(g)\cdot\chi_j'(g) = \sum_{k=1}^{r} \tau_{ij}^{(k)}\,\chi_k'(g) \qquad (\tau_{ij}^{(k)} \in C)$$

The unit of the ring X_R is the principle character $\chi_1'(g) \equiv 1$. Since the K'-characters $\chi_i'(g)$ $(i = 1,\dots,r)$ are functions of the K'-division of G, then the functions $\xi(g) \in X_R$ are functions of the K'-division of the group:

(22'.2) $\quad \xi(a) = \xi(b)$ if a and b belong to one K'-division of G.

Let us introduce the notations:

(22''.2) $\qquad X_C = X\ ; \qquad X_{I_{\mathfrak{p}}} = X_{\mathfrak{p}}$

Inasmuch as each character $\chi_i'(g)$ is an integer linear combination of absolutely irreducible characters, the ring $X_{\mathfrak{p}}$ can be considered as a subring of the ring $\hat{X}_{\mathfrak{p}}$: $\hat{X}_{\mathfrak{p}} = \{\alpha_1\hat{\chi}_1 + \dots + \alpha_s\hat{\chi}_s\}$, where $\alpha_i \in I_{\mathfrak{p}}$ $(i = 1,\dots,s)$; $\hat{\chi}_1,\dots,\hat{\chi}_s$ are characters of representations of G which are irreducible over $\hat{K}$ ($\hat{K}$ is the algebraic closure of K').

The ring $X_{\mathfrak{p}}$ has been investigated in detail by Roquette in [7].

Because $\hat{\chi}_i(g)$ is the sum of n-th roots of unity $(i = 1,\ldots,s)$, $\xi(g) \in I_{\mathfrak{p}}$ for any function $\xi \in \hat{X}_{\mathfrak{p}}$ and an arbitrary element $g \in G$.

Consequently, $\hat{X}_{\mathfrak{p}}$ is a subring of the ring A of all functions $f(g)$ $(g \in G)$ such that $f(g) \in I_{\mathfrak{p}}$. Topology can be introduced in the ring A according to the following law: $f_i \to f$ if $f_i(g) \to f(g)$ in the ring $I_{\mathfrak{p}}$ for all $g \in G$. There holds [7]

THEOREM 4.2. Any $I_{\mathfrak{p}}$-submodulus of the ring $\hat{X}_{\mathfrak{p}}$ is closed in A.

The ring $\hat{X}_{\mathfrak{p}}$ is decomposed into the direct sum of undecomposable ideals:

(23.2) $$X_{\mathfrak{p}} = B_1 + \ldots + B_q$$

The decomposition of the unit of the ring $X_{\mathfrak{p}}$ into a sum of pairwise orthogonal idempotents

(24.2) $$1 = \eta_1 + \ldots + \eta_q ; \quad \eta_i \eta_j = 0 \quad \text{if } i \neq j$$

corresponds to the decomposition (23.2).

Each of the idempotent functions $_i(g)$ $(i = 1,\ldots,a)$ takes on only two values on G: 0 and 1. Therefore, the idempotent η_i $(i = 1,\ldots,q)$ is uniquely defined by the set M_i of all elements $g \in G$ for which $\eta_i(g) = 1$ $(i = 1,\ldots,q)$.

The ideal $B_i = X_{\mathfrak{p}}\eta_i$ consists of all functions $\xi(g) \in X_{\mathfrak{p}}$ which satisfy the condition:

(25.2) $$\xi(g) = 0 \quad \text{if } g \bar{\in} M_i \quad (i = 1,\ldots,q)$$

Because of (24.2)

(26.2) $$M_1 \cup \ldots \cup M_q = G ; \quad M_i \cap M_j = \Lambda \quad \text{for } i \neq j$$

DEFINITION 4.2. We shall designate the sets $M_1,\ldots,M_q$ the $\mathfrak{p}$-divisions of the group G.

Since the character $\chi_i'(g)$ $(i = 1,\ldots,r)$ is a function of the K'-division of the group G [see THEOREM 2.1 and (21.1)], then any $\mathfrak{p}$-division M_i $(i = 1,\ldots,q)$ consists of several K'-divisions of G.

THEOREM 5.2. Every undecomposable ideal B_i contains a single ideal V_i $(i = 1,\ldots,q)$. This ideal consists of all functions $\xi \in B_i$ such that $(g) \equiv 0 \pmod{\mathfrak{p}}$ for any $g \in G$.

PROOF. The THEOREM is proved exactly as is the corresponding proposition of [7] (when B_i is an undecomposable ideal in $\hat{X}_{\mathfrak{p}}$).

Let $\xi \in B_i$. On the basis of the Euler theorem for ideals, for any natural m

$$(\xi(g))^{\varphi(\mathfrak{p}^m)} = \begin{cases} 1 \pmod{\mathfrak{p}^m} & \text{if } \xi(g) \not\equiv 0 \pmod{\mathfrak{p}} \\ 0 \pmod{\mathfrak{p}^m} & \text{if } \xi(g) \equiv 0 \pmod{\mathfrak{p}} \end{cases}$$

Therefore

$$(27.2) \qquad \lim_{m\to\infty} (\xi(g))^{\varphi(\mathfrak{p}^m)} = \eta'(g) = \begin{cases} 1 & \text{if } \xi(g) \not\equiv 0 \pmod{\mathfrak{p}} \\ 0 & \text{if } \xi(g) \equiv 0 \pmod{\mathfrak{p}} \end{cases}$$

There results from THEOREM 4.2 and (27.2) that $\eta'(g) \in B_i$.

Now let us assume that $\xi \bar{\in} V_i$. Then there exists at least one element $g \in G$ for which $\xi(g) \not\equiv 0 \pmod{\mathfrak{p}}$ and this means that the function $\eta'(g)$ is a non-zero idempotent in the ring B_i $(i = 1,\ldots,q)$ because of (27.2). Since B_i is an undecomposable ideal, then evidently $\eta' = \eta_i$. Thus

$$(28.2) \qquad \lim_{m\to\infty} \xi^{\varphi(\mathfrak{p}^m)}(g) = \eta_i(g) \qquad (\xi \bar{\in} V_i \ ; \ i = 1,\ldots,q)$$

It follows from (28.2) that there exists an inverse element $\xi' = \lim_{m\to\infty} \xi^{\varphi(\mathfrak{p}^m)-1}$ for any function $\xi \bar{\in} V_i$ $(\xi \in B_i)$. This means that V_i is an unique maximum ideal in the ring B_i $(i = 1,\ldots,q)$.

COROLLARY. **The elements** $a, b \in G$ belong to one $\mathfrak{p}$ -division of M_i if and only if for all $\xi \in X_{\mathfrak{p}}$

$$\xi(a) \equiv \xi(b) \pmod{\mathfrak{p}}$$

PROOF. If $a \in M_i$, $b \in M_j$ $(i \neq j)$, then because of (25.2), $\eta_i(a) = 1$, $\eta_i(b) = 0$ and, therefore

$$\eta_i(a) \not\equiv \eta_i(b) \pmod{\mathfrak{p}}$$

Let us establish the necessity of the condition of the THEOREM. Let $a, b \in M_i$, $\xi \in X_{\mathfrak{p}}$. Because of (23.2) the function is represented uniquely in the form

$$(29.2) \qquad \xi = \xi_1 + \ldots + \xi_q \qquad (\xi_i \in B_i \ ; \ i = 1,\ldots,q)$$

As a consequence of (25.2), $\xi_j(a) = \xi_j(b) = 0$ if $j \neq i$. This means

$$(30.2) \qquad \xi(a) = \xi_i(a) \ ; \qquad \xi(b) = \xi_i(b)$$

The set of functions $\xi \in B_i$ satisfying the condition $\xi(a) = 0$ evidently forms the ideal $I \subset B_i$. Since V_i is a unique maximum ideal in B_i by virtue of THEOREM 5.2 , then $I \equiv 0 \pmod{V_i}$. Therefore

$$\xi_i - \xi_i(a)\eta_i \in V_i$$

Hence, we conclude that

$$(31.2) \qquad \xi_i(g) - \xi_i(a)\,\eta_i(g) \equiv 0 \pmod{\mathfrak{p}}$$

for any element $g \in G$.

Putting $g = b$ in (31.2), we obtain:

$$\xi_i(b) - \xi_i(a)\ \eta_i(b) \equiv 0 \pmod{\mathfrak{p}}$$

from which, in view of (30.2), $\xi(a) \equiv \xi(b) \pmod{\mathfrak{p}}$ $(\eta_i(b) = 1)$.

The THEOREM is proved.

The element $g \in G$ is called p -regular if the order of g is mutually prime to p .

Each element $g \in G$ is represented uniquely in the form of a product of permutable factors: $g = ag'$, where a is a p -regular element and the order of g' is a power of p . The element a is called a p -regular factor of g . If g is a p -regular element, then it coincides with its p -regular factor.

Let us introduce the following relation between elements of the group G :

$g \sim g_1$ if p -regular factors of the elements g and g_1 are K' -conjugates (see DEFINITION 8.1).

Evidently this relation is reflexive, symmetric and transitive and, therefore, the set G decomposes into nonintersecting subsets of mutually equivalent elements:

(32.2) $$G = N_1 \cup \ldots \cup N_k ; \qquad N_i \cap N_j = \Lambda \quad \text{if } i \neq j$$

Each subset N_i $(i = 1,\ldots,k)$ consists of several K' -divisions of G .

Actually, if the elements of the group G are K' -conjugate, then their p -regular factors are also K' -conjugate.

It is easy to see that any subset N_i $(i = 1,\ldots,k)$ contains one and only one K' -division of the group G consisting of p -regular elements.

This means that the number of subsets N_i equals the number of such K' -divisions of the group G , into which only p -regular elements enter.

If a is a p -regular factor of the element $g \in G$ and $\xi \in X_{\mathfrak{p}}$, then the following formula holds:

(33.2) $$\xi(g) \equiv \xi(a) \pmod{\mathfrak{p}}$$

Actually, let $g = ag'$ $(ag' = g'a$; the order of g' equals $p^t)$. Since $\xi = \alpha_1\chi_1' + \ldots + \alpha_r\chi_r'$ $(\alpha_i \in I_{\mathfrak{p}})$ (see the notations (D)), then it is sufficient to prove formula (33.2) for each character χ_i' $(i = 1,\ldots,r)$.

The representation Γ_i', to which the character χ_i' $(i = 1,\dots,r)$ corresponds, induces the representation of a cyclic subgroup (g) which decomposes into the sum of one-dimensional absolutely irreducible representations. We will have for the character χ of a one-dimensional representation of the group (g):

$$\chi(ag') = \chi(a)\chi(g') \equiv \chi(a) \pmod{\mathfrak{p}}$$

because the congruence $\varepsilon' \equiv 1 \pmod{\mathfrak{p}}$ is satisfied for any root ε' of unity of degree p^t.

LEMMA 5.2. Each subset N_i is contained in one and only one $\mathfrak{p}$-division of M_j [see (26.2) and (32.2)].

PROOF. Let $g, g_1 \in N_i$ $(1 \leq i \leq k)$. Then the p-regular factors a and a_1 of the elements g and g_1, respectively, are K'-conjugate and, because of (21'.2), $\xi(a) = \xi(a_1)$ for any function $\xi \in X_{\mathfrak{p}}$. By virtue of (33.2), $\xi(g) \equiv \xi(a) \pmod{\mathfrak{p}}$ and $\xi(g_1) \equiv \xi(a_1) \pmod{\mathfrak{p}}$. Therefore

$$\xi(g) \equiv \xi(g_1) \pmod{\mathfrak{p}}$$

Applying the COROLLARY to THEOREM 5.2, we obtain that the elements g and g_1 belong to one $\mathfrak{p}$-division M_j $(1 \leq j \leq q)$.

LEMMA 6.2. If $H = (a)$ is a cyclic group of order h and $h \not\equiv 0 \pmod{p}$, then the $\mathfrak{p}$-divisions of the group H coincide with its K'-divisions.

PROOF. The group X' of absolutely irreducible characters of H is isomorphic to H.

Let us assume that χ is the generating element of the group X': $X' = (\chi)$. Let $T_1 = \{a^{s_{11}}, \dots, a^{s_{1r_1}}\}, \dots, T_m = \{a^{s_{m1}}, \dots, a^{s_{mr_m}}\}$ all be K'-divisions of H; $t_i = a^{s_{i1}} + \dots + a^{s_{ir_i}}$ is the sum of elements of a K'-division of T_i $(i = 1,\dots,m)$ in $R(H,K')$. Then the functions $\chi_i(g) = \chi^{s_{i1}}(g) + \dots + \chi^{s_{ir_i}}(g)$ $(g \in H;\ i = 1,\dots,m)$ will be K'-characters of H corresponding to irreducible representations of H over the field K'.

Actually, $\chi^r(g^\mu) = \chi^r(g) \dots \chi^r(g) = \chi^{r\mu}(g)$. This means that the characters of that K'-division of characters to which χ^r belongs are represented in the form $\chi^{r\mu_i}$, where μ_i are integers such that the mappings $\varepsilon \to \varepsilon^{\mu_i}$ are automorphisms of the field $K'(\varepsilon)$ over K'.

Let the minimum idempotents $e_1, \ldots, e_m$ of the group algebra $R(H,K')$ be expressed by the formulas:

(34.2) $$e_i = \frac{1}{h}(\lambda_{i1}t_1 + \ldots + \lambda_{im}t_m) \qquad (i = 1, \ldots, m)$$

[see (11.1) and (20.1)]. Evidently, $\frac{\lambda_{ij}}{h} \in I_{\mathfrak{p}} \quad (i,j = 1, \ldots, m)$.

Replacing the elements t_j by the characters χ_j $(j = 1, \ldots, m)$ in (34.2), we obtain m pairwise orthogonal idempotents $\eta_1, \ldots, \eta_m$ in the ring $X_{\mathfrak{p}}(H)$:

$$\eta_i = \frac{1}{h}(\lambda_{i1}\chi_1 + \ldots + \lambda_{1m}\chi_m) \qquad (i = 1, \ldots, m)$$

Since each $\mathfrak{p}$ -division of H consists of several K' -divisions of H and the number of idempotents η_i equals the number of K' -divisions of H , then the $\mathfrak{p}$ -divisions of H coincide with the K' -divisions of this group.

The LEMMA is proved.

COROLLARY. If H is a cyclic group of order $h \not\equiv 0 \pmod{p}$ and the elements $a, b \in H$, then $\chi(a) \equiv \chi(b) \pmod{\mathfrak{p}}$ for any K' -character χ of the group H , when a and b belong to one K' -division of H and there exists an irreducible K' -character χ for which $\chi(a) \not\equiv \chi(b) \pmod{\mathfrak{p}}$ if a and b are in different K' -divisions of H .

Actually, if the congruence $\chi(a) \equiv \chi(b) \pmod{\mathfrak{p}}$ is satisfied for all irreducible K' -characters χ of the group H , then the congruence $\xi(a) \equiv \xi(b) \pmod{\mathfrak{p}}$ also holds for any function $\xi \in X_{\mathfrak{p}}$. Taking into account the COROLLARY to THEOREM 5.2) and LEMMA 6.2, we arrive at the conclusion that a and b are contained in one K' -division of H .

LEMMA 7.2. Let $G = H \cdot F$ be a K' -elementary group, where $H = (a)$ is a cyclic group; F a p -group. If T is a K' -division of the group H containing a and N is a normalizer of the element a in G , $N' = F \cap N$, then the set TN' is a $\mathfrak{p}$ -division of G .

PROOF. The p -regular factors of elements of the set TN' are evidently K' -conjugate to a . On the basis of LEMMA 5.2, it follows from this fact that $TN' \subseteq M$, where M is a certain $\mathfrak{p}$ -division of the group G . This means

(35.2) $$\xi(a) \equiv \xi(g) \pmod{\mathfrak{p}} \qquad \text{if} \quad g \in TN'$$

Let the decomposition of H into a direct product of primary cyclic components have the form

(36.2) $$H = H_1 \times \ldots \times H_k$$

By virtue of (36.2), we obtain the decomposition of the element a :

(37.2) $$a = a_1 \dots a_k \qquad (H_i = (a_i) \; ; \; i = 1,\dots,k)$$

Let N_i be the normalizer of the element a_i in G ; $N_i' = F \cap N_i$;

$$Q_i = (H_1 \times \dots \times H_{i-1} \times H_{i+1} \dots \times H_k)\cdot N_i' \quad (i = 1,\dots,k)$$

Evidently, $N_i = H_i \times Q_i$ $(i = 1,\dots,k)$. It is easy to see that

(38.2) $$N = N_1 \cap \dots \cap N_k$$

Let $\chi_1^{(i)}$ (the principal character), $\chi_2^{(i)}, \dots, \chi_{q_i}^{(i)}$ be all the irreducible K' -characters of the group H_i $(i = 1,\dots,k)$. For each i $(i = 1,\dots,k)$ there exists a character $\chi_j^{(i)}$ $(1 < j \leq q_i)$ such that $\chi_j^{(i)}(a_i) \not\equiv 0 \pmod{\mathfrak{p}}$. Actually, if $\chi_j^{(i)}(a_i) \equiv 0 \pmod{\mathfrak{p}}$ $(j = 2,\dots,q)$, then

(39.2) $$\sum_{j=1}^{q_i} \chi_j^{(i)}(a_i) \equiv 1 \pmod{\mathfrak{p}}$$

since $\chi_1^{(i)}(a_i) = 1$. On the other hand

(40.2) $$\sum_{j=1}^{q_i} \chi_j^{(i)}(a_i) = 0$$

because $\sum_{j=1}^{q_i} \chi_j^{(i)}(a_i) = \sum_{j=1}^{h_i} \chi_j^{(i)}(a_i)$, where $\chi_1^{(i)}, \dots, \chi_{h_i}^{(i)}$ are all the irreducible characters of the group H_i $(i = 1,\dots,k)$ (see $(\widetilde{\mathrm{IV}})$, § 1) .

There results from (39.2) and (40.2) the contradictory congruence: $1 \equiv 0 \pmod{\mathfrak{p}}$.

Let us select the character $\chi_{j_i}^{(i)}$ $(1 < j_i \leq q_i)$ of the group H_i $(i = 1,\dots,k)$ satisfying the condition: $\chi_{j_i}^{(i)}(a_i) \not\equiv 0 \pmod{\mathfrak{p}}$. According to LEMMA 4.2, a K' -character $\chi_{j_i}^{(i)}$ of the group H_i is induced by the K' -character $\chi_{j_i}'^{(i)}$ of the group G :

(41.2) $$\chi_{j_i}'^{(i)}(g) = \begin{cases} 0 & \text{if } g \bar{\in} N_i \\ \chi_{j_i}^{(i)}(b_i) & \text{if } g \in N_i \text{ and } g = b_i c \quad (b_i \in H_i \; ; \; c \in Q_i) \end{cases} \qquad (i = 1,\dots,k)$$

Let us consider the K' -character $\chi'(g)$:

(42.2) $$\chi'(g) = \chi_{j_1}'^{(1)}(g) \dots \chi_{j_k}'^{(k)}(g)$$

As a consequence of (37.2), (41.2) and (42.2)

$$\chi'(a) = \chi_{j_1}^{(1)}(a_1) \dots \chi_{j_k}^{(k)}(a_k)$$

Because $\mathfrak{p}$ is a prime ideal and $\chi_{j_i}^{(i)}(a_i) \not\equiv 0 \pmod{\mathfrak{p}}$ $(i = 1,\ldots,k)$, we obtain that

(42'.2) $$\chi'(a) \not\equiv 0 \pmod{\mathfrak{p}}$$

From (42.2), (41.2) and (38.2) there follows the equality:

(43.2) $$\chi'(g) \equiv 0 \quad \text{if} \quad g \in N$$

Comparing (42'.2) and (43.2), we arrive at the relation:

(44.2) $$\chi'(g) \not\equiv \chi'(a) \pmod{\mathfrak{p}} \quad \text{if} \quad g \bar{\in} N$$

Now, let us take an element $g \in G$ satisfying the conditions: $g \in N$; $g \bar{\in} TN'$. Then g is represented in the form:

(45.2) $$g = a^t n' \quad \text{where} \quad a^t \bar{\in} T, \quad n' \in N'$$

Because of the COROLLARY to LEMMA 6.2, there results from (45.2) that there exists an irreducible K'-character χ of the group H such that

(46.2) $$\chi(a) \not\equiv \chi(a^t) \pmod{\mathfrak{p}}$$

On the basis of the COROLLARY to LEMMA 4.2, the character χ is induced by a certain irreducible K'-character $\tilde{\chi}'$ of the group G. Hence

(47.2) $$\tilde{\chi}'(g) = \tilde{\chi}'(a^t n') \equiv \tilde{\chi}'(a^t) \pmod{\mathfrak{p}}$$

Since $\tilde{\chi}'(a^t) = \chi(a^t)$, then by virtue of (46.2) and (47.2)

(48.2) $$\tilde{\chi}'(g) \not\equiv \tilde{\chi}'(a) \pmod{\mathfrak{p}} \quad \text{if} \quad g \in N \quad \text{and} \quad g \bar{\in} TN'$$

Because of the COROLLARY to THEOREM 5.2, the relations (35.2), (44.2) and (48.2) prove that TN' is a $\mathfrak{p}$-division of the group G.

LEMMA 8.2. Let G be an arbitrary finite group; H a subgroup of order m of the group G; $X(G)$ and $X(H)$ integer rings of K'-characters of the groups G and H, respectively (see (22''.2)); $\tilde{R}$ a ring of all functions $f(g)$ in the group G which satisfy the conditions:

1) $f(g)$ takes on values in the field $K'' \supseteq K'$ and is a function of a K'-division of the group G.

2) $f(g)$ induces the function $f_H(g) \in X(H)$ in the subgroup H.

If I is an ideal in the ring $X(H)$, then the set I^* of functions

(49.2) $$\xi^*(g) = \frac{1}{m} \sum_{c \in G} \xi(c^{-1}gc) \quad (\xi(c^{-1}gc) = 0 \ \text{if} \ c^{-1}gc \bar{\in} H)$$

induced by the functions $\xi(g) \in I$, forms an ideal in the ring $\tilde{R}$ (it is evident that $X(G) \subseteq \tilde{R}$).

PROOF. Because of (49.2), $\xi^* - \eta^* = (\xi - \eta)^* \in I^*$ $(\xi, \eta \in I)$. Let $f(g) \in \tilde{R}$, $\eta(g) \in I$ and $f_H(g)$ be a function induced by $f(g)$ in H $(f(g) = f_H(g)$ for all $g \in H)$. Then as a consequence of (49.2)

$$f(g)\eta^{*}(g) = \frac{1}{m} \sum_{c \in G} f(g)\eta(c^{-1}gc) = \frac{1}{m} \sum_{c \in G} f(c^{-1}gc)\eta(c^{-1}gc) = (f_H(g)\cdot\eta(g))^{*}$$

where $f_H(g)\eta(g) \in I$ since $\eta(g) \in I$.

If R is a ring which is a group of linear forms over C : $R = (u_1,\ldots,u_m)$, then we will denote a ring of linear combinations $\{\lambda_1 u_1 + \ldots + \lambda_m u_m\}$, where $\lambda_i \in I_{\mathfrak{p}}$, by means of $R_{\mathfrak{p}}$ and the basis elements u_i $(i = 1,\ldots,m)$ are multiplied exactly as in the ring R (see the notations (D)).

LEMMA 9.2. Let R be a ring with the unit whose elements form a group of linear forms over C with respect to addition; let T be an ideal in R . If $T_{\mathfrak{p}} = R_{\mathfrak{p}}$ for all primes p , then $T = R$.

PROOF. Let us consider R as a subring of $R_{\mathfrak{p}}$. The maximum number of elements of the ring $R_{\mathfrak{p}}$ which are linearly independent over $I_{\mathfrak{p}}$ equals the rank of R over C and the maximum number elements in the ring $T_{\mathfrak{p}}$ which are linearly independent over $I_{\mathfrak{p}}$ equals the rank of T over C . Since $T \quad = R \quad$, then the ranks of the free Abelian groups R and T coincide.

Hence, if $u_1,\ldots,u_m$ is the basis of T over C , then the following relation holds:

(50.2) $$\gamma\cdot 1 = \alpha_1 u_1 + \ldots + \alpha_m u_m \qquad (\alpha_i \in C\ ,\ \gamma \in C\ ,\ i = 1,\ldots,m)$$

We conclude from the equality $R_{\mathfrak{p}} = T_{\mathfrak{p}}$ that in $R_{\mathfrak{p}}$

(51.2) $$1 = \beta_1 u_1 + \ldots + \beta_m u_m \qquad (\beta_i \in I_{\mathfrak{p}}\ ;\ i = 1,\ldots,m)$$

Comparing (50.2) and (51.2), we arrive at the relations:

(52.2) $$\frac{\alpha_i}{\gamma} = \beta_i \in I_{\mathfrak{p}} \qquad (i = 1,\ldots,m)$$

Formulas (52.2) are valid for all primes p $(p \equiv 0 \pmod{\mathfrak{p}})$. A rational number which is a -adic integer for any prime p belongs to the ring C .

This means $1 = \frac{\alpha_1}{\gamma} u_1 + \ldots + \frac{\alpha_m}{\gamma} \in T$ and $T = R$.

The LEMMA is proved.

Let G be an arbitrary finite group; B_i an undecomposable ideal in $X_{\mathfrak{p}}(G)$; V_i a maximum ideal in B_i ; η_i an idempotent generating the block B_i in $X_{\mathfrak{p}}(G)$; M_i a $\mathfrak{p}$-division of the group G , corresponding to B_i ; $a_i \in M_i$ a p-regular element; $D_j \subseteq M_i$ the set of all elements of G whose p-regular factors are K'-conjugate to a_i ; $N_{K'}(a_i)$ the K'-normalizer of the element a_i in G ; F_i a Silow p-subgroup of the group $N_{K'}(a_i)$; $G_i' = (a_i)F_i$ is a K'-elementary sugroup of the group G ; N_i the normalizer of a_i in G_i' ; $N_i' = F_i \cap N_i$; T_i is a K'-division of the cyclic group (a_i) containing the element a_i $(i = 1,\dots,q)$.

By virtue of LEMMA 7.2, the set T_iN_i' is a $\mathfrak{p}$-division of the group G_i'. Therefore, there exists the idempotent

$$\eta_i'(g) = \begin{cases} 1 & \text{if } g \in T_iN_i' \\ 0 & \text{if } g \bar{\in} T_iN_i' \end{cases} \quad (g \in G_i') \tag{53.2}$$

in the ring $X_{\mathfrak{p}}(G_i')$.

DEFINITION 5.2. The ideal E_i generated by the idempotent η_i' in $X_{\mathfrak{p}}(G_i')$ will be called an elementary block corresponding to the ideal B_i $(i = 1,\dots,q)$.*

THEOREM 6.2. The ideal B_i consists of all functions $\xi^* \in X_{\mathfrak{p}}(G)$ induced by the functions $\xi \in E_i$ (E_i is an elementary block corresponding to B_i) $(i = 1,\dots,q)$.

PROOF. Let $\xi^* \in E_i$. Then the induced function $\xi^* \in X_{\mathfrak{p}}(G)$ is expressed, according to (49.2), by the formula:

$$\xi^*(g) = \frac{1}{(G_i':1)} \sum_{c \in G} \xi(c^{-1}gc) \tag{54.2}$$

where $\xi(c^{-1}gc) = 0$ if $c^{-1}gc \bar{\in} G_i'$. By virtue of LEMMA 8.2, the set E_i^* of functions ξ^* $(\xi \in E_i)$ generates an ideal in $X_{\mathfrak{p}}(G)$.

The block E_i consists of all functions $\xi \in X_{\mathfrak{p}}(G')$ satisfying the condition: $\xi(g) = 0$, if $g \bar{\in} T_iN'$. This means that $\xi(g) = 0$ if $c^{-1}gc \in T_iN_i'$ for all $c \in G$, i.e., if the p-regular factors of the elements g and a_i are not K'-conjugate $(\xi \in E_i)$.

Hence

$$\xi^*(g) = 0 \qquad \text{if } g \bar{\in} D_j \tag{54'.2}$$

Since $D_j \subseteq M_i$, then because of (54'.2), $\xi^*(g) = 0$ when $g \bar{\in} M_i$. From this we conclude that $E_i^* \subseteq B_i$ because B_i consists of all functions $\tilde{\xi} \in X_{\mathfrak{p}}(G)$ such that $\tilde{\xi}(g) = 0$ if $g \bar{\in} M_i$.

Putting $g = a_i$ and $\xi = \eta_i'$ in (54.2), we obtain:

$$\eta_i'^*(a_i) = \frac{1}{(G_i':1)} \sum_{c \in G} \eta_i'(c^{-1}gc) \qquad (\eta_i'(c^{-1}gc) = 0 \tag{55.2}$$

when $c^{-1}a_ic \bar{\in} G_i'$). By virtue of (53.2), formula (55.2) can be written as:

* The block E_i depends on the selection of the p-regular element a_i in the $\mathfrak{p}$-division M_i $(i = 1,\dots,q)$.

$$(55'.2) \qquad \eta_i'^{*}(a_i) = \frac{1}{(G_i':1)} \sum_{c\in G} \eta_i'(c^{-1}gc)$$

where

$$\eta_i'(c^{-1}gc) = \begin{cases} 0 & \text{if } c^{-1}a_ic \bar{\in} T_iN_i' \\ 1 & \text{if } c^{-1}a_ic \in T_iN_i' \end{cases}$$

Because a_i is a p-regular element, $c^{-1}a_ic \in T_iN_i'$ if and only if $c^{-1}a_ic \in T$, i.e., when $c \in N_{K'}(a_i)$.

On the basis of the last remark, we obtain from (55'.2):

$$(56.2) \qquad \eta_i'^{*}(a_i) = \frac{(N_{K'}(a_i):1)}{(G_i':1)}$$

Since G' contains the Silow p-subgroup of the group $N_{K'}(a_i)$, then because of (56.2)

$$(57.2) \qquad \eta_i'^{*}(a_i) \not\equiv 0 \pmod{\mathfrak{p}}$$

According to THEOREM 5.2, the congruence $\tilde{\xi}(g) \equiv 0 \pmod{\mathfrak{p}}$ (g is an arbitrary element of the group G) is satisfied for the function $\tilde{\xi}(g) \in V_i$.

We now conclude from (57.2) that $\eta_i'^{*} \bar{\in} V_i$. This means

$$(58.2) \qquad E_i^{*} = B_i$$

(Because of THEOREM 5.2, V_i is a single maximum ideal in the ring B_i.)

The THEOREM is proved.

Taking into account (54'.2) and (57.2), we arrive at the relations:

$$(59.2) \qquad \eta_i'^{*}(g) = 0 \quad \text{if} \quad g \bar{\in} D_j \ ; \quad \eta_i'^{*}(a_i) \not\equiv 0 \ (\text{mod} \quad)$$

The equality $D_j = M_i$ results from the existence of the function $\eta_i'^{*}(g)$ satisfying the conditions (59.2) and from the inclusion $D_j \subseteq M_i$ on the basis of the COROLLARY to THEOREM 5.2.

Hence, there holds

THEOREM 7.2. Each $\mathfrak{p}$-division of the group G consists of all elements of G having K'-conjugate p-regular factors.

This means the number of undecomposable ideals in the direct decomposition of the ring $X_{\mathfrak{p}}$ equals the number of K'-divisions of G which contain p-regular elements.

Comparing (23.2) and (58.2), we obtain the direct decomposition:

$$(60.2) \qquad X_{\mathfrak{p}} = E_1^{*} + \dots + E_q^{*}$$

where $E_1, \dots, E_q$ is a system of elementary blocks corresponding to the undecomposable direct components $B_1, \dots, B_q$ of the ring $X_{\mathfrak{p}}$.

The decomposition of (60.2) is valid for any prime p $(p = 0 \pmod{\mathfrak{p}})$.

Let X be an integer ring of K'-characters of the group G (see (22.2)); $H_1,\ldots,H_s$ all the K'-elementary subgroups of the group G; X_i the integer ring of the K'-characters of the group H_i; X_i^* the set of all functions of the ring X induced by functions from X_i $(i = 1,\ldots,s)$. Then the decomposition (not direct) holds:

$$X = X_1^* + \ldots + X_s^* \tag{61.2}$$

Actually, by virtue of LEMMA 8.2, the sum $\tilde{X} = X_1^* + \ldots + X_s^*$ is an ideal in X. On the basis of (60.2), $\tilde{X}_{\mathfrak{p}} = X_{\mathfrak{p}}$ for all primes p. Applying LEMMA 9.2, we obtain that $\tilde{X} = X$.

The following theorem, generalizing the Brauer theorem [5], results from (61.2):

THEOREM 8.2. Each character of the group G corresponding to the representation of the group G over the field K' of characteristic zero is an integer linear combination of K'-characters induced by K'-characters of K'-elementary subgroups of the group G.

DEFINITION 6.2. Let us call the elements of the integer ring of K'-characters X generalized K'-characters of the group G.

THEOREM 9.2. The function $f(g)$, prescribed on the group G with values in the field $K'' \supseteq K'$, is a generalized K'-character of the group G if and only if the following conditions are satisfied:

1) $f(g)$ is a function of the K'-division of the group G.

2) $f(g)$ induces a generalized K'-character of the K'-elementary subgroup of the group G in each subgroup.

PROOF. Let $\tilde{R}$ be the ring of all functions $f(g)$ with values in the field $K'' \supseteq K'$ satisfying conditions 1) and 2) of the THEOREM. It is evident that $X \subseteq \tilde{R}$. On the basis of LEMMA 8.2, the submodulus $R_0 \subseteq X$ generated by K'-characters induced by K'-characters of the K'-characters of the K'-elementary subgroups of the group G is an ideal in the ring R. By virtue of THEOREM 8.2, $R_0 = X$. This means that $R_0 = X = R$.

The THEOREM is proved.

THEOREM 9.2. generalizes the fundamental result of the work [6].

THEOREM 10.2. The integer ring of K'-characters of a finite group, where K' is an arbitrary field of characteristic zero which is not decomposable in a direct sum of ideals.

PROOF. The integer ring X of K'-characters of the group G can be considered as a subring of the ring $\hat{X}$, where $\hat{X}$ — an integer ring of characters of the group G corresponding to the irreducible representations of G over a field $\hat{K}$ — is the algebraic closure of the field K'.

Let $\hat{X}_{\hat{K}}$ be a ring of absolutely irreducible characters of the group G over the field $\hat{K}$. The elements $\hat{X}_{\hat{K}}$ are all possible linear combinations

$$\alpha_1\hat{\chi}_1 + \dots + \alpha_s\hat{\chi}_s$$

($\alpha_i \in \hat{K}$; $i = 1,\dots,s$; $\hat{\chi}_1,\dots,\hat{\chi}_s$ are absolutely irreducible characters of the group G).

Let us assume that $C_1,\dots,C_s$ is a class of conjugate elements of the group G ; h_i the order of the class C_i ; $a_1,\dots,a_s$ the system of representatives of classes $C_1,\dots,C_s$; h the order of G. Let us consider the elements $e_i \in \hat{X}_{\hat{K}}$:

$$e_i = \frac{h_i}{h}\sum_{j=1}^{s} \chi_j(a_i)\chi_j \qquad (i = 1,\dots,s) \tag{62.2}$$

By virtue of (III) (§ 1)

$$e_i(g) = \begin{cases} 0 & \text{if } g \bar{\in} C_i \\ 1 & \text{if } g \in C_i \end{cases} \tag{63.2}$$

This means that $e_i,\dots,e_s$ are idempotents of the algebra $\hat{X}_{\hat{K}}$.

Because of (63.2), $e_i(g)e_j(g) = 0$ if $i \neq j$; $e_1 + \dots + e_s = 1$ (the unit of the algebra $\hat{X}_{\hat{K}}$ is the principal character $\chi_1(g) \equiv 1$).

Since the number of idempotents e_i equals the rank of the algebra $\hat{X}_{\hat{K}}$ over $\hat{K}$, then $\hat{X}_{\hat{K}}$ is a semisimple algebra over $\hat{K}$.

Each idempotent $e \in \hat{X}_{\hat{K}}$ is represented in the form of a sum of certain of the idempotents $e_1,\dots,e_s$:

$$e = e_{i_1} + \dots + e_{i_t} \quad (1 \leq t \leq s) \tag{64.2}$$

On the basis of (64.2) and (62.2)

$$e = \left(\frac{h_{i_1}}{h}\chi_1(a_{i_1}) + \dots + \frac{h_{i_t}}{h}\chi_1(a_{i_t})\right)\chi_i + \dots = \frac{h_{i_1} + \dots + h_{i_t}}{h}\chi_1 + \dots$$

Since $h_1 + \dots + h_s = h$, then $\dfrac{h_{i_1} + \dots + h_{i_t}}{h}$ C (C is the ring of rational numbers) if and only if $h_{i_1} + \dots + h_{i_t} = h_1 + \dots + h_s = h$.

But in this case $e_{i_1} + \dots + e_{i_t} = e_1 + \dots + e_s = 1$.

Thus, the ring $\hat{X}$ contains only trivial idempotents. This means the ring $X \subseteq \hat{X}$ is undecomposable into a direct sum of ideals.

The THEOREM is proved.

Exactly as the theorem inverse to the Brauer theorem is proved in [13], let us prove a theorem inverse to THEOREM 8.2.

Let us say that the system of subgroups $M = \{H_\alpha\}$ of the group G has the property (B) if each K'-character of the group G is an integer linear combination of K'-characters induced by K'-characters of the subgroup H_α .

Continuing the reasoning on which the proof of THEOREM 9.2 was based, it is easy to show that the system of subgroups $M = \{H_\alpha\}$ has the property (B) if and only if any function of the K'-division of the group $f(g)$ ($f(g) \in K'' \supseteq K'$) which induces a generalized K'-character of the subgroup H_α on each subgroup, is a generalized K'-character of the group G .

Hence, there results that by replacing any subgroup H_α in the system $M = \{H_\alpha\}$ satisfying the condition (B) with a subgroup $H'_\alpha \supseteq H_\alpha$ or a subgroup conjugate to H_α and also by adding new subgroups to the system M (discarding subgroups from the system so that the property (B) would be retained), we again obtain a system having the property (B) .

Two systems of subgroups M and M' satisfying the condition (B) and obtained from each other by the transformation mentioned, we shall agree to call equivalent.

THEOREM 11.2. If the system of subgroups $M = \{H_\alpha\}$ of the group G has the property (B) , then there exists a subgroup $H_\alpha \in M$ for any K'-elementary subgroup $E' \subseteq G$, which contains a subgroup conjugate to E' .

Hence, to the accuracy of equivalence, the system of K'-elementary subgroups of G is a single set of subgroups having the property (B) .

PROOF. Let E' be a K'-elementary subgroup generated by a p-regular element a and a p-subgroup $P' \subseteq N_{K'}(a)$ ($N_{K'}(a)$ is the

K' -normalizer of the element a in G). Let us adjoin P' to the Silow p -subgroup $P \subseteq N_{K'}(a)$ and let us show that a subgroup $H_\alpha \in M$ is found which contains a subgroup conjugate to the group $E = \{a, P\}$.

Let ψ_{ij} be an arbitrary irreducible K' -character of a certain subgroup H_i . According to (1'.2)

$$(65.2) \qquad \psi^*_{ij}(a) = \frac{h}{h'.h_a} \sum_{g \in C_a \cap H} \psi_{ij}(g) = \sum_{t=1}^{r} \frac{h \cdot h_t}{h'.h_a} \psi_{ij}(b_t) = \sum_{t=1}^{r} \alpha_{ijt} \psi_{ij}(b_t)$$

where h is the order of G ; h' the order of H_i ; C_a the class of conjugate elements of the group G generated by the element a ; $b_1, \ldots, b_r$ a system of representatives of classes of conjugate elements $C'_1, \ldots, C'_r$ of the group H_i into which the intersection $C_a \cap H$ is decomposed; h'_t is the order of the class of conjugate elements C'_t of the group H $(t = 1, \ldots, r)$.

Since $\frac{h}{h_a} = n$ is the order of the normalizer $N_G(b_t)$ of the element b_t in G and $\frac{h'}{h'_t} = n'$ is the order of the normalizer $N_{H_i}(b_t)$ in H_i $(N_{H_i}(b_t) = N_G(b_t) \cap H_i)$, then α_{ijt} is the index of $N_{H_i}(b_t)$ in $N_G(b_t)$.

α_{ijt} can also be considered as the index of the K' -normalizer $N^{(i)}_{K'}(b_t)$ of the element b_t in H_i in the K' -normalizer $N_{K'}(b_t)$ of the same element in the group G .

Actually, because the set of elements $g \in G$ satisfying the condition $g^{-1} b_t g = b_t^\mu$ (μ is a fixed number) forms a contiguous class of the group G by means of $N_G(b_t)$, the order of $N_{K'}(b_t)$ is ns , where s is the number of elements of the cyclic group (b_t) K' -conjugate to b_t and in exactly the same way the order of $N^{(i)}_{K'}(b_t)$ is $n's$.

This means $(N_{K'}(b_t) : N^{(i)}_{K'}(b_t)) = \frac{ns}{n's} = \frac{n}{n'} = \alpha_{ijt}$.

Since the system of subgroups $M = \{H_i\}$ has, according to the condition, the property (B) , then $\sum_{i,j} \gamma_{ij} \psi^*_{ij} = 1$ (γ_{ij} are integers), from which by virtue of (65.2), there results that for certain i, j, t

$$(66.2) \qquad \alpha_{ijt} \not\equiv 0 \pmod p$$

It follows from (66.2) that $N^{(i)}_{K'}(b_t)$ contains a Silow p -subgroup Q of the group $N_{K'}(b_t)$ and this means that the K' -elementary subgroup is

$$E' = \{b_t, Q\} \subseteq N^{(i)}_{K'}(b_t) \subseteq H_i$$

As a consequence of the inclusion $b_t, a \in C_a$ $(b_t = c^{-1}ac ; c \in G)$ we conclude that $c^{-1}Pc$ is a Silow p-subgroup of the group $N_{K'}(b_t)$. This means that $d^{-1}(c^{-1}Pc)d = Q$ for a certain element $d \in N_{K'}(b_t)$, from which $(cd)^{-1}Ecd = E' \subseteq H_i$.

The THEOREM is proved.

I am grateful to IA. B. Lopatinskii and I. R. Shafarevich for a number of valuable remarks.

Uzhgerod Oct. 17, 1956

References

1. M. VAN DER WAERDEN: Modern Algebra. 1947
2. G. FROBENIUS: Theory of characters and group representations. 1937
3. G. FROBENIUS, I. SCHUR: Ueber die reelen Darstellungen der endlichen Gruppen. Berliner Ber., 657, 186 - 203 (1906)
4. I. SCHUR: Arithmetische Untersuchungen ueber endliche Gruppen linearer Substitutionen. Berliner Ber., 657, 164 - 184 (1906)
5. R. BRAUER: On Artin's L-series with general group characters. Ann. of Math., 48(1), 502 - 514 (1947)
6. R. BRAUER: A characterization of the characters of groups of finite order. Ann. of Math. 57(2), 358 - 377 (1953)
7. P. ROQUETTE: Arithmetic investigations of the character rings of a finite group. Journ. reine und angew. Math., 190, 148-168 (1952)
8. S. D. BERMAN: On the theory of representations of finite groups. DAN USSR, 86(6), 885-888 (1952)
9. S. D. BERMAN: On the isomorphism of centers of group rings of p-groups DAN USSR, 91(2), 185 -187 (1953)
10. S. D. BERMAN: p-adic rings of characters. DAN USSR, 106(4), 583-586 (1956)
11. I. D. ADO: On the theory of linear representations of finite groups. Math. Sbornik, 36(78), 25-30 (1955)
12. S. D. BERMAN: Number of irreducible representations of a finite group over an arbitrary field. DAN USSR, 106(5), 767-769 (1956)
13. J. A. GREEN: On the converse to a theorem of R. Brauer. Proc. Cambr. Phil. Soc., 51(1), 237 - 239 (1955)
14. ASAKO KEIZO: Einfacher Beweis eines Brauerschen Satzes ueber Gruppencharaktere. Proc. Japan Acad., 31(8), 501 - 503 (1955)
15. R. BRAUER; J. TATE: On characters of finite groups. Ann. of Math., 62(1), 1 - 7 (1955)
16. R. BRAUER: Applications of induced characters. Amer. Journ. Math. 69(4), 709 - 716 (1947)
17. S. PERLIS, G. L. WALKER: Abelian group algebras of finite order. Trans. Amer. Math. Soc., 68(3), 420 - 426 (1950)
18. S. D. BERMAN: On the equation $x^m = 1$ in an integer group ring. Ukr. Math. Journal, 7(3), 253 - 261 (1955)
19. S. D. BERMAN: Groups all of whose representations are monomials. DAN UkrSSR, No. 6, 539 - 542 (1957)